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Applicability of Intimate Mapping in Digital Image Sources

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***Abstract*— After 1922 definition of contraction mapping, its applicability and usefulness were worked upon by several researchers through their various research papers. Different fields of engineering, science and technology could not remain untouched by its use. The notion of compatible mapping and compatible mapping of type (*A*) is further transformed with the concept of intimate mapping.**

The development of operator theory was brought forward by the idea of fixed points impending upon contraction mappings. Apart from general metric space, metric spaces like b-metric space, G-metric space, F-metric space and digital space enhanced the applicability of fixed-point theory. We wish to put forward the notion of intimate mapping in digital metric space through some consequences on it.

***Index Terms*— digital topology, digital metric space, compatible mapping, intimate mapping.**

I. INTRODUCTION

In digital topology, the topological characters of image pictures are featured by some array of elements. A specific arrangement of non-negative numbers in a digital topological space is denoted by a digital image. A digital image can be fragmented into its constituents to analyse its various features. This method is nothing but image processing in a digital metric space. By using the method of digital image fragmentation, a digital picture can be improved through tracking, coding, counting, thinning, contour filling, etc. Thus, the applicability of digital technology appeared as a solid technique of image processing. The primary basis of image processing is connectedness and adjacency relations.

Rosenfield, A. [1] [2], visualized digital topology by proposing digital continuous function. The three-dimensional image thinning suggested by Kong [4] and works done by Boxer, L. [8] [9] on digital continuous mappings enriched the concept. Ege, Karaca, and many other researchers [10] [11] [12] [13] endowed their contributions to the field of digital topology. Applicability of fixed points in this field was improvised by the authors in references [10] [11].

Mathematical topology was contributed with the notion of common fixed points, compatibility and type (*A*) compatibility by Jungck, G. [3] [5]

Sahu et. al., defined the concept of intimate mapping in general metric spaces which is nothing but the generalization of type (A) compatible mappings [6]. It has been shown by several authors as in references; [7-16] that in digital technology, digital metric space can be defined to strengthen its application in certain fields like image processing, remote sensing, and many more. Employing coincidence point theorem in b-metric space, the

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author in ref. [15] has shown a similar result in digital topology.

II. PRELIMINARIES

In n-dimensional Euclidean space, a subset Y of set Z^n of lattice points, for any integer $n \geq 0$, with some adjacency relation k in Y is known as a digital image, denoted by (Y, k) [1,2].

Definition 1: [14] For any two fixed points, $u = (u_{1,} u_{2,} \dots \dots \dots . u_{\mathrm{n}})$, $v = (v_{1,} v_{2,} \dots \dots \dots . v_{\mathrm{n}}) \in Z^n$ and positive integers k, n; $1 \leq k \leq n$, we define u, v as k-adjacent if for at most k, we obtain
$|u_i - v_i| = 1$, for all i and $|u_j - v_j| \neq 1$, $u_j = v_j$ for all other indices$j$.

Thus, we have For $n = 1$, any two points u and v in Z have 2-adjacent, if $|u - v| = 1$. A symbolic diagram is shown in figure below.

2-Adjacency

For two distinct points u and v that have a difference in at most one coordinate, we have 8-adjacent in Z^2(for $n = 2$) and 26-adjacent Z^3(for $n = 3$). However, if the points are 8-adjacent and differ in exactly, one co-ordinate then the two points u and v in Z^2(for $n = 2$) are 4-adjacent and they are 18-adjacent, if the points are 26-adjacent having at most two distinct coordinates.

This can be generalized for n-dimensional digital image as shown below [16].

$$\text{For } 1 \leq k \leq n, \text{ we have: } p(k, n) = \sum_{i=n-a}^{n-1} 2^{n-i} \binom{n}{i}, \text{where } \binom{n}{i} = \frac{n!}{i!(n-i)!} \qquad \dots (1)$$

For a non-empty set $Y \in Z^n$, s.t. $1 \leq k \leq n$ and $p = p(k, n)$. Then, $p(k, n)$ is known as a digital image having k-adjacency [8]. And, the pair (Y, k) is the n-D digital image [15].

Definition 2: [4] [14] Let us define a digital interval $[l, m]_z$, $\forall\, l, m \in Z$ and $l < m$ as:

$$[l, m]_z = \{z \in Z: l \leq z \leq m\}.$$

Definition 3: [16] Any point of Z^n, k-adjacent to u is known as k-neighbour of $u \in Z^n$ if for $k \in \{2,4,8,6,18,26\}$ and $n \in \{1,2,3\}$, we get

$$P_k(u) = \{v: v \text{ is } k\text{-adjacent to } u\}$$

$P_k(u)$ is a k-neighbourhood to k-neighbour of u. u and v are k-neighbours if v is k-adjacent to u.

Definition 4: [16] A k-connected digital image of $Y \in Z^n$ is detailed for each $l, m \in Y$ if $\exists$ a set $\{l_0, l_1, l_2, l_3, l_{4,} \dots \dots \dots, l_r\}$, in such a way that $l = l_0$ and $m = m_r$ & l_i and l_{i+1} are k-neighbours, $\forall i = 0,1,2,3,4, \dots \dots \dots, r-1$.

Definition 5: [16] Assume digital images (Y, k_0) and (N, k_1) of Z^{n_0} and Z^{n_1}, respectively and a mapping $T: Y \rightarrow N$. Then, we obtain:

1. T is (k_0, k_1)-continuous if k_0-connected subsets of Y correspond to k_1-connected to N.
2. T is (k_0, k_1)-continuous *iff* images of k_0-adjacent to Y are k_1-adjacent in N else, images of a_0-adjacent of Y are coinciding. That is, l_0, l_1 are k_0-adjacent to Y then, $T(l_0)$ and $T(l_1)$ are k_1-adjacent in N or, $T(l_0) = T(l_1)$.
3. T is called (k_0, k_1)-isomorphism if

a. T is(k_0, k_1)-continuous
b. T is one-to-one and onto
c. T^{-1} is (k_0, k_1)-continuous.

So, we get $Y \cong_{(k_0, k_1)} \mathrm{N}$.

Definition 6: [16] For a set Y a digital k-path from $l \rightarrow m$ is defined as a $(2, \mathrm{k})$-continuous function $\mathrm{T}: [0, r]_z \rightarrow Y$ s. t. $T(0) = l$ and $T(\mathrm{r}) = m$. The set Y is called k-path connected in the set (Y, k) if there is a k-path for any two points.

Definition 7: [12] Consider a digital image (Y, k) and a mapping $\theta: (Y, k) \times (Y, k) \rightarrow Z^n$ with following conditions:

1. $\theta(y_1, y_2)$ is non-negative
2. $\theta(y_1, y_2) = 0 \; iff \; y_1 = y_2$
3. $\theta(y_1, y_2) = \theta(y_2, y_1)$
4. $\theta(y_1, y_3) \leq \theta(y_1, y_2) + \theta(y_2, y_3)$, $\forall y_1, y_2, y_3 \in Y$.

Then, θ is defined as a digital metric and (Y, θ, k) as digital metric space.

Definition 8: [16] Suppose we have any sequence $< y_n >$ on (Y, θ, k), where (Y, θ, k) be a digital metric space. Then, we say $< y_n >$ as a Cauchy sequence, if for each $\varepsilon > 0 \ \exists \ c \in N$, we have

$$\theta(y_r, y_n) < \varepsilon, \text{ for each } r, n > c.$$

Notion 1: [14] Let $< y_n >$ is a Cauchy sequence on a digital metric space (Y, θ, k) then, we obtain

$$y_r = y_n, \forall\, r, n > c \in N$$

Definition 9: [16] A sequence $< y_n >$ being a Cauchy sequence on (Y, θ, k), where (Y, θ, k) be a digital metric space, is called convergent on Y, if for each $\varepsilon > 0 \ \exists \ c \in N$, such that

$$\theta(y_n, y_0) < \varepsilon, \forall\, n > c.$$

Notion 2: [14] Digital metric space (Y, θ, k) is said to be complete if every Cauchy sequence $< y_n >$ converses to any point $y_0 \in \mathrm{Y}$ in it. So, (Y, θ, k) is complete.

Definition 10: [16] A digital contraction defined on a self-map $g: (Y, \theta, k) \to (Y, \theta, k)$ for any digital contraction constant $\rho \in [0,1)$ as:

$$\theta(\mathrm{gy}, \mathrm{gz}) \leq \rho\theta(\mathrm{y}, \mathrm{z}), \forall y, z \in Y$$

is always digitally continuous.

Notion 3: [11] For any digital contraction defined on a self-map $g: (Y, \theta, k) \to (Y, \theta, k)$ in a digital metric space (Y, θ, k) on Euclidean metric Z^n there exists a unique fixed point $y_0 \in Y$ such that $g(y_0) = y_0$.

Definition 11: [3] Mappings $g, h: (Y, \theta, k) \to (Y, \theta, k)$ are said to be compatible mappings if and only if,

$\lim_{n\to\infty} \theta\big(hg(\mathrm{y}_n), gh(\mathrm{y}_n)\big) = 0$,

where $< y_n >$ is a sequence such that $\lim_n g(\mathrm{y}_n) = \lim_n h(\mathrm{y}_n) = a$, for some $a \in \mathrm{Y}$.

Notion 4: It is obvious that, a weakly commuting pair of mappings are compatible but compatibility does not imply that pairs are weakly commuting.

For example, consider the mappings $g(y) = 2y^3$ and $h(y) = y^3$, $\mathrm{y} \in \mathrm{Y}$. Then,

$$|g(\mathrm{y}) - h(\mathrm{y})| = |2y^3 - y^3| = |y^3| \to 0$$

$$\Leftrightarrow |gh(\mathrm{y}) - hg(\mathrm{y})| = |2(y^3)^3 - (2y^3)^3| = 6|y^9| \to 0$$

So, g and h are compatible.

But they are not weakly commuting as, $|gh(\mathrm{y}) - hg(\mathrm{y})| \nleq |g(\mathrm{y}) - h(\mathrm{y})| \ \forall\, y \in R$

Definition 12: [5] Mappings $g, h: (Y, \theta, k) \to (Y, \theta, k)$ are said to be compatible mappings of type (A) if,

(i). $\lim_{n\to\infty} \theta\big(hg(\mathrm{y}_n), g^2(\mathrm{y}_n)\big) = 0$

(ii). $\lim_{n\to\infty} \theta\big(gh(\mathrm{y}_n), h^2(\mathrm{y}_n)\big) = 0$

where $< y_n >$ is a sequence such that $\lim_n g(\mathrm{y}_n) = \lim_n h(\mathrm{y}_n) = a$, for some $a \in \mathrm{Y}$.

Definition 13: [6] For a non-empty set $Y \in Z^n$ and digital image (Y, k), any two mappings g and h in digital metric space (Y, θ, k) such that $g, h: (Y, \theta, k) \to (Y, \theta, k)$ are known as digitally intimate mappings,

(i). denoted by digitally h-intimate, if $\omega\, \theta\big(hg(\mathrm{y}_n), h(\mathrm{y}_n)\big) \leq \omega\, \theta\big(gg(\mathrm{y}_n), g(\mathrm{y}_n)\big)$ and

(ii). denoted by digitally g-intimate, if $\omega\, \theta\big(gh(\mathrm{y}_n), g(\mathrm{y}_n)\big) \leq \omega\, \theta\big(hh(\mathrm{y}_n), h(\mathrm{y}_n)\big)$,

where $< y_n >$ is a sequence such that $\lim_n g(\mathrm{y}_n) = \lim_n h(\mathrm{y}_n) = a$, for some $a \in \mathrm{Y}$ and $\omega = limit\ sup.$ or $limit\ inf.$

Notion 5: A pair of any two mappings g and h of a digital metric space (Y, θ, k) onto (Y, θ, k) itself are type (A) digitally compatible then they are digitally g, h-intimate.

Converse of Notion 5 is not followed, in general.

Theorem 2.1: BFPT in Digital Metric Space [14]:

Let (Y, θ, k) be a digital metric space and Banach type digital contraction on Euclidean metric $Z^n$is defined on a self-map $g: (Y, \theta, k) \to (Y, \theta, k)$ for any constant $\rho \in [0,1)$ as:

$$\theta(\mathrm{gy}, \mathrm{gz}) \leq \rho\theta(\mathrm{y}, \mathrm{z}), \forall y, z \in Y$$

Then, there exists a unique fixed point $y_0 \in Y$, which gives $g(y_0) = y_0$.

III. Main Result

Let us introduce a notion on intimate mappings as a contraction:

Notion 6: Consider two digitally h-intimate mappings G and H on a digital metric space (Y, θ, k) such that

$$H(a) = G(a) = b, \qquad b \in Y$$

Then, we get

$$\theta(Hb, b) \leq \theta(Gb, b).$$

Now, we establish a result on common fixed point for pairs of intimate mappings G and H as follows:

Theorem 3.1: Consider a non-empty set $Y \in Z^n, n \in N$ and a digital image (Y, k) with an adjacency relation k in Y. Let (Y, θ, k) be a digital metric space having mappings G and H such that,

$$H(Y) \subseteq G(Y)$$

$$\theta(Hx, Hy) \leq \rho\theta(Gx, Gy), \forall x, y \in Y.$$

Where, $\rho \in [0,1)$. Then G and H have a common fixed point.

Proof: Since $H(Y) \subseteq G(Y)$, so we can always construct a sequence $< x_n >$ such that

$$H(x_n) \subseteq G(x_{n+1}).$$

Now,

$$\theta(Gx_{n+1}, Gx_n) = \theta(Hx_n, Hx_{n-1})$$

$$\leq \rho\,\theta(Gx_n, Gx_{n-1}) = \rho\theta(Hx_{n-1}, Hx_{n-2})$$

$$\leq \rho^2\theta(Gx_{n-1}, Gx_{n-2})$$

Continuing this process, we get,

$$\theta(Gx_{n+1}, Gx_n) \leq \rho^n d(Gx_1, Gx_0), \rho \in [0,1)$$

So as, $n \to \infty$, we have $\theta(Gx_{n+1}, Gx_n) \to 0$, then by Banach contraction principle, we obtain

$$H(x_n) = G(x_{n+1}) \to z \in (Y, \theta, k)$$

Hence, z is a unique common fixed point of G and H.

Theorem 3.2: Let (Y, θ, k) be a digital metric space and $H: Y \to Y$ be a self mapping such that,
$\theta(Hx, Hy) \leq \rho\theta(Gx, Gy)$, $\forall\, x, y \in Y$ and $\rho \in [0,1)$.
Then, H has a unique fixed point in Y.

Proof: If we consider, $G = I$ (an identity mapping) in Theorem 3.1, then, the result of this theorem is a particular case of our Theorem 3.1.

Corollary 3.1: Consider a non-empty set $Y \in Z^n, n \in N$ and a digital image (Y, k) with an adjacency relation k in Y. Let (Y, θ, k) be a digital metric space having strictly increasing mappings $G, H: (Y, \theta, k) \to (Y, \theta, k)$ such that,

I. $H(Y) \subseteq G(Y)$
II. $\theta(Hx, Hy) \leq \rho\theta(Gx, Gy), \forall x, y \in Y$, where $\rho \in [0,1)$.

Then, the mappings G and H are compatible.

Proof: By using Theorem 3.1, we can show that the two self-mappings G and H have a common fixed point. Hence, the result is obvious as two strictly increasing self-mappings having a common fixed point are compatible as [3]:

$$G(x_n), H(x_n) \to z$$

$$\Rightarrow \lim_{n\to\infty} \theta\big(HG(\mathrm{y}_n), GH(\mathrm{y}_n)\big) = 0, \text{ for } z \in \mathrm{Y}.$$

Thus, we have come across the result which may enrich the theory of a common fixed point in digital metric space.

IV. Conclusions

In dealing with digital metric space, sets are generally integral valued so the Cauchy sequences obtained may have constant values. Thus, it may or may not sound good. However, regarding connected digital images, the metric using contraction assures the shortest path length. In regular practice, storage of the data is a measure problem now-a-day. So, compression, reduction in size and thinning of digital image sources are the need of the hour. Thus, by making use of common fixed-point theorem in digital image processing, the desired image without much redundancy can be reinforced.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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