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# New Bounds on Eigen Values of Normalized Signed Graph

B.Prashanth[1], Madhusudan K V[2] and Nagendra Naik K[3]
[1]Department of Mathematics, JSS Science and Technology University, Mysuru-570 006, India
Email: bprassy@gmail.com
[2]Department of Mathematics, A.T.M.E College of Engineering, Bannur Road, Mysuru, India.
Email: kvmadhu13@gmail.com
[3]Department of Mathematics, Atria Institute of Technology, Hebbal, Bangalore-560024, India.
Email: nagendranaiknk@gmail.com

***Abstract*— In relation to vertex *i* of a signed graph Γ, let $d$ be the degree of any vertex, $\rho_1$ be the least eigen value of *Normalized Laplacian* matrix of signed graph $\rho_1$ and $T_i^+$ indicates the number of positive (balanced) triangles passing through $i$ respectively. We prove that if Γ be a complete signed graph then $\rho_1 \geq \min\left\{\frac{4T_i^+}{d^2} - 1\right\}$.**

***Index Terms*— Mathematics Subject Classification: 05C22, 05C50, 15A18, signed graphs, marked graphs, balance, Switching.**

## I. Introduction

Your goal is to simulate the usual appearance of papers in a Conference Proceedings or Journal Publications. We are requesting that you follow these guidelines as closely as possible.

Readers should see [5] for graph theory phrases and notations since only simple and finite graphs are taken into account in this article.

We forego going into specifics of signed graph $\Gamma(G, \sigma)$ where $G$ is the graph without sign and $\sigma$ is the mapping between set of edges and the set $\{+, -\}$ and a list of previous findings in this paper in favor of presenting our contribution we refer to [6],[7],[14] for those who require a more thorough strategy.

The vertices degree of $\Gamma(G, \sigma)$ and $G$ are same because each vertex's degree can be determined using the formula

$$d = d^+ + d^-.$$

If two vertices are adjacent each other, the entry $a_{ij}$ in the adjacent matrix A(Γ) is 1 along with the edge's sign; if they are not, the entry is 0 and Laplacian of a signed graph Γ is the matrix $M = I - K^{-1/2}A(\Gamma)K^{1/2}$

### *A. Normalized Laplacian Matrix of signed graph*

The Normalized Laplacian matrix was first introduced by F.R.K. Chung [16]. Grossman researched the lower bounds of the Normalized Laplacian, and Chen et al. described its features in their papers [10] and [3].

Normalized LaplacianMatrix of a graph with sign Γ having vertices $p$ and $q$, is given by

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$$L_{pq} = \begin{cases} 1, & if\ p = q\ and\ d_p \neq 0 \\ -\sigma(pq)\dfrac{1}{\sqrt{d_p d_q}}, & if\ p\ and\ q\ are\ adjacent \\ 0 & otherwise. \end{cases}$$

Eigen values of Normalized Laplacian matrix of $\Gamma$ , with n vertices be $0 \leq p_1 \leq p_2 \leq p_3 \ldots\ldots\ldots\ldots p_{n-1}$.
Also $L = I - K^{-1/2}M(\Gamma)K^{-1/2}$ where $K$ is diagonal Matrix.

*B. Notation*

We denote the degree of a vertex $p \in v(\Gamma)$ by $d_p$, $d_p^+$and $d_p^-$ are the edges with positive and negative signs incident with p respectively. $p \sim$ q follows $p$ and q are adjacent.
$p \overset{+}{\sim} q$ (respectively $p \overset{-}{\sim} q$) represents the positive (resp. negative) edge between the vertices $p$ and $q$ and the value $m_p$ denotes the average degree of $p$'s neighbours. For the (possibly distinct) vertices $p$ and $q$, $l_{pq}$ is the notation to represent the number of their neighbours in common (i.e., those connected to both of them by any edge), $l_{pq}^+$ stands for the quantity of neighbours that are connected to $p$ by a positive edge and to $q$ by any edge, and the number of neighbours that share a positive edge with both of them is $l_{pq}^{++}$ .
The writings of the remaining ones are all identical.
As we are aware, $L = K^{-1/2}\ L\ K^{-1/2}$ Where $K$ is invertible.

If eigen value $\rho_1$ corresponds to eigen function $h$ of L, then $t = K^{1/2}\ h$ ,
$t_j = K^{1/2}\ \ _j$

$$\rho_1 = \inf \frac{\sum (h(p) - \sigma(p,q)h(q))^2}{\sum_q h^2(p)d_p}$$

Where $d_p$ represents the degree of the vertex $p$.
For a vertex $q$

$$(1-\rho_1)h(q) = \frac{1}{d_p}\sum_{p\sim q}\sigma(p,q)h(p)$$

Where, $p \sim q$ means $p$ and $q$ vertices are adjacent. Let $p_{1,}\, p_{2,} \ldots\ldots. p_{m+1}$ be adjacent vertices sequence. For convenience set $(1-\rho_1) = \gamma$ .

## II. Results

**Proposition 1.** For any signed graph $\Gamma(G\,,\sigma)$ ,

$$\rho_1 \geq \frac{4T_i^+ - d_i m_i - \sum_{j\sim i}\sigma(i\,,j)d_j x_j}{d_i^2 - \sum_{j\sim i}\sigma(i\,,j)d_j x_j}$$

**Proof.** Let $x$ be an eigenvector with respect to $\rho_1$ and $x_i$ , the biggest coordinate in modulus. We can assume that $x_i = 1$without compromising generality.

We have

$$(1-\rho_1)x_i = \tfrac{1}{d_i} = \sum_{u\sim i}\sigma(u\,,i)x_u$$

$$\gamma x_i di = \sum_{u\sim i}\sigma(u\,,i)x_u$$

$$\gamma d_i = \sum_{\substack{u \overset{+}{\sim} j \\ u \nsim i}} x_u + \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{+}{\sim} j}} x_u + \sum_{\substack{u \overset{+}{\sim} i \\ u \overset{-}{\sim} j}} x_u - \sum_{\substack{u \overset{-}{\sim} i \\ u \nsim j}} x_u - \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{-}{\sim} j}} x_u - \sum_{\substack{u \overset{-}{\sim} i \\ u \overset{+}{\sim} j}} x_u \qquad (1)$$

Let $x_j$ be another coordinate,

$$\gamma x_j d_i = \sum_{\substack{u \overset{+}{\sim} j \\ u \nsim i}} x_u + \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{+}{\sim} i}} x_u + \sum_{\substack{u \overset{+}{\sim} j \\ u \overset{-}{\sim} i}} x_u - \sum_{\substack{u \overset{-}{\sim} j \\ u \nsim i}} x_u - \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{-}{\sim} i}} x_u - \sum_{\substack{u \overset{-}{\sim} j \\ u \overset{+}{\sim} j}} x_u \qquad (2)$$

By summing (1) and (2) we get

$$\gamma d_i + \gamma x_j d_j = \sum_{\substack{u \overset{+}{\sim} i \\ u \nsim j}} x_u + \sum_{\substack{u \overset{+}{\sim} j \\ u \nsim i}} x_u - \sum_{\substack{u \overset{-}{\sim} i \\ u \nsim j}} x_u - \sum_{\substack{u \overset{-}{\sim} j \\ u \nsim i}} x_u + 2\sum_{\substack{u \overset{+}{\sim} i \\ u \overset{+}{\sim} j}} x_u - 2\sum_{\substack{u \overset{-}{\sim} i \\ u \overset{-}{\sim} j}} x_u$$

$$\gamma\left(d_i + d_j x_j\right) \le (d_i^+ - l_{ij}^+) + (d_j^+ - l_{ij}^+) + \left(d_i^- - l_{ij}^-\right) + \left(d_j^- - l_{ij}^-\right) + 2l_{ij}^{++} + 2l_{ij}^{--}$$

$$\gamma\left(d_i + d_j x_j\right) \le d_i + d_j - 2l_{ij}^{+-} - 2l_{ij}^{-+}$$

When summed up holistically as $j \overset{-}{\sim} i$

$$\gamma(d_i d_i^- + \sum_{j \overset{-}{\sim} i} d_j x_j) \le d_i d_i^- + \sum_{j \overset{-}{\sim} i} d_j - 2\sum_{j \overset{-}{\sim} i} T_{ij}^+ \qquad (3)$$

Similarly (1) - (2) gives ,

$$\gamma d_i - \gamma x_j d_i = \sum_{\substack{u \overset{+}{\sim} i \\ u \nsim j}} x_u - \sum_{\substack{u \overset{-}{\sim} i \\ u \nsim j}} x_u - \sum_{\substack{u \overset{+}{\sim} j \\ u \nsim i}} x_u + \sum_{\substack{u \overset{-}{\sim} j \\ u \nsim i}} x_u + 2\sum_{\substack{u \overset{+}{\sim} i \\ u \overset{-}{\sim} j}} x_u - 2\sum_{\substack{u \overset{-}{\sim} i \\ u \overset{+}{\sim} j}} x_u$$

$$\gamma\left(d_i - d_j x_j\right) \le (d_i^+ - l_{ij}^+) + (d_i^- - l_{ij}^-) + \left(d_j^+ - l_{ji}^+\right) + \left(d_j^- - l_{ji}^-\right) + 2l_{ij}^{+-} + 2l_{ij}^{-+}.$$

$$\gamma\left(d_i - d_j x_j\right) \le d_i + d_j - 2l_{ij}^{++} - 2l_{ij}^{--}.$$

When summed up holistically as $j \overset{+}{\sim} i$

$$\gamma(d_i d_i^+ - \sum_{j \overset{+}{\sim} i} d_j x_j) \le d_i d_i^+ + \sum_{j \overset{+}{\sim} i} d_j - 2\sum_{j \overset{+}{\sim} i} T_{ij}^+ \qquad (4)$$

(3) + (4) implies,

$$\gamma\left(d_i^2 - \sum_{j \sim i} \sigma(i, j) d_j x_j\right) \le d_i^2 + \sum_{j \sim i} \sigma(i, j) d_j - 4\sum_{j \sim i} T_{ij}^+$$

$$\gamma\left(d_i^2 - \sum_{j \sim i} \sigma(i, j)\, d_j x_j\right) \le d_i^2 + d_i m_i - 4T_i^+$$

$$\gamma \leq \frac{d_i^2 + d_i m_i - 4T_i^+}{d_i^2 - \sum_{j\sim i} \sigma(i\,,j) d_j x_j}$$

$$\rho_1 \geq \frac{4T_i^+ - d_i m_i - \sum_{j\sim i} \sigma(i\,,j) d_j x_j}{d_i^2 - \sum_{j\sim i} \sigma(i\,,j) d_j x_j}$$

**Corollary 2.** Let $\Gamma$ be a complete signed graph then

$$\rho_1 \geq \min\left\{\frac{4T_i^+}{d^2} - 1\right\}. \tag{5}$$

**Proof.** Since, $\Gamma$ is a complete signed graph,

$$\sum_{j\sim i} \sigma(i,j)\, x_j\, d_j = d_i \sum_{j\sim i} \sigma(i\,,j) x_j \tag{6}$$

$$\sum_{J\sim i} \sigma(i,j)\, x_j d_j = \sum_{\substack{j \overset{+}{\sim} i \\ j \nsim k}} x_j + \sum_{\substack{j \overset{+}{\sim} i \\ u \overset{+}{\sim} k}} x_j + \sum_{\substack{j \overset{+}{\sim} i \\ j \overset{-}{\sim} k}} x_j - \sum_{\substack{j \overset{-}{\sim} i \\ j \nsim k}} x_j - \sum_{\substack{j \overset{-}{\sim} i \\ j \overset{-}{\sim} k}} x_j - \sum_{\substack{j \overset{-}{\sim} i \\ j \overset{+}{\sim} k}} x_j$$

$$\leq (d_i^+ - l_{ik}^+) + l_{ik}^{++} + l_{ik}^{--} + (d_k^- - l_{ki}^-) + 2l_{ik}^{+-} + 2l_{ik}^{-+}$$
$$\leq (d_i - l_{ik})$$

$$-d_i \sum_{j\sim i} \sigma(i\,,j) x_j \geq -d_i^2 + d_i l_{ik}.$$

From (6)

$$\sum_{j\sim i} \sigma(i\,,j) x_j\, d_j \geq -d_i^2 + d_i l_{ik}.$$

From Proposition 1

$$\rho_1 \geq \frac{4T_i^+ - d_i m_i - \mathrm{d}_i^2 + d_i l_{ik}}{d_i^2 - \mathrm{d}_i^2 + d_i l_{ik}}$$

Since $l_{ik} = d_i$

$$d_i m_i = d_i^2$$

$$\rho_1 \geq \frac{4T_i^+}{d_i^2} - 1$$

Taking the minimum over all vertices, we arrive at (5).

## III. Conclusions

In this paper we have given an elegant proof of a new upper bound of Normalized signed graph and we believe that this new approach will pave way for further research in this area.

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