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On the Jameson-Schneider and Apostol Theorems

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***Abstract*— We show that the Jameson-Schneider's theorem for arithmetic functions implies the Apostol theorem, which allows to obtain recurrence relations for several functions of interest in number theory.**

***Index Terms*— Arithmetic functions, Recurrence relations, Bell polynomials, Triangular numbers.**

I. Introduction

Jameson-Schneider [1] proved the following result for any sequence $\{s(n)\}$:

$$\frac{q\,\frac{d}{dq}\prod_{n=1}^{\infty}(1-q^n)^{s(n)}}{\prod_{j=1}^{\infty}(1-q^j)^{s(j)}} = -\sum_{n=1}^{\infty}\left(\sum_{d|n} s(d)\,d\right)q^n, \tag{1}$$

which in Section 2 allows us to deduce a Theorem of Apostol [2, 3] for arithmetic functions [4]. In Section 3, we show that equation (1) implies the recurrence relations obtained in [5] for $r_j(n)$, the number of representations of n as a sum of j squares , for $t_k(n)$, the number of representations of n as a sum of k triangular numbers [6, 7], and also for $p_k(n)$, the number of color partitions of n [8-10].

II. Apostol theorem

If the property

$$\prod_{n=1}^{\infty}(1-q^n)^{s(n)} = \sum_{j=0}^{\infty} R(j)\,q^j, \qquad R(0)=1, \tag{2}$$

holds, then equation (1) implies that

$$\sum_{n=0}^{\infty} n\,R(n)\,q^n = -\sum_{n=0}^{\infty} c(n)\,q^n, \qquad c(n)=\sum_{j=1}^{n}\left(\sum_{d|j} s(d)\,d\right)R(n-j),$$

that is

$$n\,R(n) = -c(n) = -\sum_{j=1}^{n}\left(\sum_{d|j} s(d)\,d\right)R(n-j). \tag{3}$$

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Then, for the case $s(n) = \frac{1}{n} f(n)$, where $f(n)$ is an arithmetic function, the expressions (2) and (3) reproduce the Apostol theorem [2, 3]:
If

$$\prod_{n=1}^{\infty} (1-q^n)^{\frac{f(n)}{n}} = \sum_{j=0}^{\infty} R(j)\, q^j, \quad R(0) = 1,$$

then

$$nR(n) = -\sum_{j=1}^{n} \left(\sum_{d|j} f(d)\right) R(n-j). \tag{4}$$

III. APPLICATIONS OF (2) AND (3)

a) If $(n) = k\,(-1)^n$, then from (2),

$$\prod_{n=1}^{\infty} (1-q^n)^{(-1)^n k} = \prod_{n=1}^{\infty} \left[\frac{1-q^{2n}}{1-q^{2n-1}}\right]^k = \sum_{j=0}^{\infty} t_k(j)\, q^j, \quad t_k(0) = 1.$$

Thus equation (3) gives the following recurrence relation [5]:

$$n\, t_k(n) = -k \sum_{j=1}^{n} \left(\sum_{d|j} (-1)^d\, d\right) t_k(n-j). \tag{5}$$

b) If $(n) = k$, then from (2),

$$\prod_{n=1}^{\infty} (1-q^n)^k = \sum_{j=0}^{\infty} p_k(j)\, q^j.$$

Hence equation (3) implies the recurrence property [5, 11, 12]:

$$n\, p_k(n) = -k \sum_{j=1}^{n} \sigma(j)\, p_k(n-j), \tag{6}$$

involving the sum of divisors function [2, 4]. For the case $k = 24$, the expression (6) generates a recurrence relation for the Ramanujan's tau-function [13, 14]:

$$n\, \tau(n+1) = -24 \sum_{j=1}^{n} \sigma(j)\, \tau(n+1-j), \tag{7}$$

whose solution can be expressed in terms of complete Bell polynomials [15, 16]:

$$\tau(n+1) = \frac{1}{n!} B_n\big(-24 \cdot 0! \cdot \sigma(1), -24 \cdot 1! \cdot \sigma(2), -24 \cdot 2! \cdot \sigma(3), \dots, -24 \cdot (n-1)! \cdot \sigma(n)\big), \quad n \ge 0. \tag{8}$$

c) If $s(n) = \frac{k}{2}\,(3-(-1)^n)$, then from (2):

$$\prod_{n=1}^{\infty} (1-q^n)^{\frac{k}{2}(3-(-1)^n)} = \prod_{n=1}^{\infty} (1-q^{2n-1})^2\, (1-q^{2n})^k = \sum_{j=0}^{\infty} (-1)^j\, r_k(j)\, q^j. \tag{9}$$

Thus, equation (3) gives the recurrence relation [5]:

$$nr_k(n) = -2k \sum_{j=1}^{n} (-1)^j\, j \left(\sum_{odd\ d|j} \frac{1}{d}\right) r_k(n-j), \tag{10}$$

where it was used the identity

$$\sum_{d|j} s(d)\, d = \frac{k}{2} \sum_{d|j} (3-(-1)^d)\, d = 2k\, j \sum_{odd\ d|j} \frac{1}{d}. \tag{11}$$

Remark 1: In [5], the following expression can be found:

$$\prod_{n=1}^{\infty} \frac{1-q^{nk}}{1+q^{n}} = \sum_{j=0}^{\infty} (-1)^j \, r_k(j) \, q^j, \tag{12}$$

whose comparison with equation (9) implies the connection

$$\prod_{n=1}^{\infty} \frac{1-q^{n}}{1+q^{n}} = \prod_{n=1}^{\infty} (1-q^{2n-1})^2 \, (1-q^{2n}), \tag{13}$$

which is valid because it is equivalent to the following known identity [17]:

$$(q;q^2)_\infty \, (-q;q)_\infty = 1. \tag{14}$$

Remark 2: For $k = -1$, the result (6) implies the recurrence relation for the partition function [18]:

$$n\,p(n) = \sum_{j=1}^{n} \sigma(j)\, p(n-j), \tag{15}$$

and for $k = 1$, equation (6) gives the recurrence relation deduced by Robbins [19] and Osler-Hassen-Chandrupatla [20]:

$$na_n = -\sum_{j=1}^{n} \sigma(j)\, a_{n-j}, \qquad a_0 = 1, \qquad \sigma(0) = 0, \qquad n \geq 1, \tag{16}$$

where

$$a_j = \begin{cases} 0, & j \neq \frac{m}{2}(3m+1), \\ & \\ (-1)^m, & j = \frac{m}{2}(3m+1), \end{cases} \qquad m = 0, \ \pm 1, \ \pm 2, \ldots \tag{17}$$

that is

$$a_j = \begin{cases} 1, & j = 0, 5, 7, 22, 26, 51, 57, 92, 100, 145, 155, \ldots \\ -1, & j = 1, 2, 12, 15, 35, 40, 70, 77, 117, 126, 176, \ldots \\ 0, & otherwise \end{cases} \tag{18}$$

d) If $s(n) = \frac{1}{2}\,[1-(-1)^n\,]$, then equation (2) generates the following property (see [17]):

$$\prod_{n=1}^{\infty} (1-q^n)^{\frac{1-(-1)^n}{2}} = \prod_{n=1}^{\infty} (1-q^{2n-1}) = \sum_{n=0}^{\infty} (-1)^n q^n \, p_{OD}(n), \tag{19}$$

where $p_{OD}(n)$ is the number of partitions of the integer *n* into parts which are both odd and distinct. Hence equation (3) implies the following Robbin's recurrence relation [19]:

$$n\,p_{OD}(n) = -\sum_{j=1}^{n} (-1)^j \left(\sum_{odd\ d|j} d\right) p_{OD}(n-j). \tag{20}$$

e) If $s(n) = \frac{(-1)^n - 1}{2}$, then from equation (2) (See [17])

$$\prod_{n=1}^{\infty} (1-q^n)^{\frac{(-1)^n-1}{2}} = \prod_{n=1}^{\infty} (1-q^{2n-1})^{-1} = \sum_{n=0}^{\infty} q^n \, p_D(n), \tag{21}$$

where $p_D(n)$ is the number of partitions of the integer *n* with only distinct parts. Then, equation (3) gives the recurrence relation [19, 22]:

$$n\,p_D(n) = \sum_{j=1}^{n} \left(\sum_{odd\ d|j} d\right) p_D(n-j). \tag{22}$$

Then, we have the Euler's result $p_D(n) = p_O(n)$, the number of partitions of the integer *n* employing only odd parts (See [17]).

REFERENCES

[1] M. Jameson, R. P. Schneider, Combinatorial applications of Möbius inversion, Proc. of Amer, Math. Soc. 142(9) (2014), 2965-2971.

[2] T. M. Apostol, Introduction to analytic number theory, Springer-Verlag, New York, (1976).
[3] R. Sivaraman, J. López-Bonilla and O. Salas-Torres, Apostol theorem applied to Möbius and Euler totient functions, submitted (2023).
[4] R. J. McCarthy, Introduction to arithmetical functions, Springer-Verlag, New York, (1986).
[5] G. E. Andrews, S. Kumar Jha and J. López-Bonilla, Sums of squares, triangular numbers and divisor sums, J. of Integer Sequences, 26 (2023), Article 23.2.5
[6] E. Grosswald, Representations of integers as sums of squares, Springer-Verlag, New York, (1985).
[7] C. J. Moreno and S. S. Wagstaff Jr, Sums of squares of integers, Chapman & Hall/CRC, Boca Raton, Fl, USA, (2006).
[8] J. López-Bonilla and J. Yaljá Montiel-Pérez, On the Gandhi's recurrence relation for colour partitions, Sarcouncil J. of Multidisciplinary, 1(1) (2021), 24-25.
[9] J. López-Bonilla and M. Morales-García, Sum of divisors function in terms of colour partitions, Studies in Nonlinear Sci., 7(1) (2022), 4-5.
[10] J. López-Bonilla, A. Lucas-Bravo and O. Marín-Martínez, On the colour partitions p_r (n), Comput. Appl. Mat. Sci., 6(2) (2021), 35-37.
[11] J. M. Gandhi, Congruences for p_r (n) and Ramanujan's τ-function, Amer. Math. Monthly, 70(3) (1963), 265-274.
[12] O. Lazarev, M. Mizuhara and B. Reid, Some results in partitions, plane partitions, and multipartitions, Summer 2010 REU Program in Maths. at Oregon State University, Aug 13, 2010.
[13] S. Ramanujan, On certain arithmetical functions, Trans. Camb. Phil. Soc., 22(9) (1916), 159-184.
[14] R. Roy, Elliptic and modular functions from Gauss to Dedekind to Hecke, Cambridge University Press (2017).
[15] R. Sivaraman, J. D. Bulnes and J. López-Bonilla, Complete Bell polynomials and recurrence relations for arithmetic functions, European J. of Theoretical and Applied Sci., 1(3) (2023), 167-170.
[16] R. Sivaraman, H. N. Núñez-Yépez and J. López-Bonilla, Ramanujan's tau-function in terms of Bell polynomials, submitted (2023).
[17] W. P. Johnson, An introduction to q-analysis, Am. Math. Soc., Providence, Rhode Island, USA, (2020).
[18] J. D. Bulnes, T. Kim, J. López-Bonilla and S. Vidal-Beltrán, On the partition function, Proc. Jangjeon Math. Soc., 26(1) (2023), 1-9.
[19] N. Robbins, Some identities connecting partition functions to other number theoretic functions, Rocky Mountain J. Math., 29(1) (1999), 335-345.
[20] T. J. Osler, A. Hassen and T. R. Chandrupatla, Surprising connections between partitions and divisors, The College Maths. J., 38(4) (2007), 278-287.
[21] J. López-Bonilla and S. Vidal-Beltrán, On the Jha and Malenfant formulae for the partition function,Comput. Appl. Math. Sci., 6(1) (2021), 21-22.
[22] N. Robbins, On partition functions and divisor sums, J. Integer Sequences, 5 (2002), Article 02.1.4