

Gallai Signed Graphs and Anti-Gallai Signed Graphs

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Abstract— In this paper we introduced the new notions Gallai and anti-Gallai signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterizations of these notions. Further, we presented some interesting switching equivalent characterizations.

Index Terms— Mathematics Subject Classification: 05C22, 05C12, Signed graphs, Balance, Switching, Gallai Signed graph, Anti-Gallai Signed Graph.

I. INTRODUCTION

For standard terminology and notation in graph theory we refer Harary [4] and Zaslavsky [27] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

The Gallai Graph $\mathcal{GL}(G)$ of a Graph $G = (V, E)$ is the Graph whose Vertex Set $V(\mathcal{GL}(G)) = E(G)$; two distinct vertices e_1 and e_2 are adjacent in $\mathcal{GL}(G)$ if e_1 and e_2 are incident in G , but do not span a triangle in G (see [6]). In fact, this concept was introduced by Gallai [3] in his examination of comparability Graphs and this notion was recommended by Sun [25]. The author Sun wasted Gallai Graphs $\mathcal{GL}(G)$ to characterize an amusing class of perfect Graphs. Gallai Graphs are also wasted in the Polynomial Time Algorithm to determinate complete Bipartite $K_{(1,3)}$ free perfect Graphs by the authors Chvatal and Sbihi [2].

The Anti-Gallai Graph $\mathcal{AGL}(G)$ of a Graph $G = (V, E)$ is the Graph whose Vertex Set $V(\mathcal{AGL}(G)) = E(G)$; two distinct vertices e_1 and e_2 are adjacent in $\mathcal{AGL}(G)$ if e_1 and e_2 are incident in G and lie on a triangle in G (see [6]). Equivalently, the Anti-Gallai Graph $\mathcal{AGL}(G)$ is the complement of Gallai Graph $\mathcal{GL}(G)$ in the line Graph $L(G)$. We can easily observe that the Gallai Graphs $\mathcal{GL}(G)$ and Anti-Gallai Graphs $\mathcal{AGL}(G)$ are the spanning subgraphs of the Line Graph $L(G)$.

A signed graph $S = (G, \sigma)$ is graph with each edge labeled by a sign (i.e., + or -). Clearly, S is a mapping from $E(G)$ to $\{+, -\}$ (e.g., see Harary [5], Sampathkumar [11], Zaslavsky [26, 27]); the set $\{+, -\}$ may either treated simply as a set of signs called colors, when σ is regarded as a 2-edge coloring of G or as the involutory Group $\{-1, +1\}$ demarcated by a multiplication table in which σ is preserved as a estimation of the edges of S . For given Signed Graph $S = (G, \sigma)$, the edge set $E(S)$ could be classified as $E(S) = E^+(S) \cup E^-(S)$, where $E^+(S)$ ($E^-(S)$) denote the set of all edges marked by + (-). The Sign of a cycle in S is the multiplication of the Signs of all edges on the Cycle. For more new notions on signed graphs refer the papers (See [8-10, 13-24]).

A marking of S is a Function $\zeta : V(G) \rightarrow \{+, -\}$. Given a Signed Graph S , one can effortlessly express a marking ζ of S as given below:

For any vertex $v \in V(S)$,

$$\zeta(V) = \prod_{uv \in E(S)} \sigma(uv)$$

the pattern ζ of S is known as Canonical Marking of S . S is then called the marked signed graph.

The subsequent are the important results about Balance, the succeeding being a more progressive form of the principal. Note that in a bipartition of a set $V = V_1 \cup V_2$, the disjoint sub-sets may be empty.

Theorem 1. A Signed Graph S is Balanced if and only if any of the subsequent corresponding conditions are contented:

- (i) It's Vertex Set has a Bi-partition $V = V_1 \cup V_2$ such that every +ve edge joins vertices in V_1 or in V_2 , and every -ve (Negative) edge connects a vertex in V_1 and a vertex in V_2 (Harary [5]).
- (ii) There exists a marking ζ of its Vertices such that very edge uv in G satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [11]).

Switching a Signed Graph S with respect to the assumed Sign means to attain another Signed Graph $S = S_\zeta(S)$ from S_ζ by altering the Sign of every of its Edge $x = uv$ for which $\zeta(u) = -\zeta(v)$. The subsequent Signed Graph $S_\zeta(S)$ is known as Switched Signed Graph. A Signed Graph S switches to alternative Signed Graph S' written $S \sim S'$ every time there exists a Sign such that $S' \cong S_\zeta(S)$. Hence, if $S \sim S'$, we intend to say that S and S' are Switching Equivalent (see [1]).

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are known to be weakly isomorphic (see [12]) or Cycle Isomorphic (see [26]) if there subsists an Isomorphism: $\emptyset : G_1 \rightarrow G_2$ so that the Sign of each Cycle Z in S_1 matches to the Sign of $\emptyset(Z)$ in S_2 . The subsequent outcome is well identified (see [26]):

Theorem 2. (T. Zaslavsky [26]) "Two Signed Graphs S_1 and S_2 with the identical underlying Graph are switching equivalent if and only if they are cycle isomorphic".

II. GALLAI SIGNED GRAPH OF A SIGNED GRAPH

The Gallai Signed Graph $\mathcal{GL}(S) = (\mathcal{GL}(G), \sigma')$ is a Signed Graph and $\mathcal{GL}(G)$ is a Gallai Graph and sign of any $e_1 e_2 \in E(\mathcal{GL}(S))$ is the multiplication of the signs of the corresponding edges in $S = (G, \sigma)$. For some signed graph S_1 such that $\mathcal{GL}(S_1) \cong S$, then S is called the Gallai signed graph.

Signed graphs can be classified into two types namely, balanced and unbalanced signed graphs. The signed graph $S = (G, \sigma)$ is balanced or unbalanced, we have $\mathcal{GL}(S)$ is always balanced in either of the cases.

Theorem 3. Let $S = (G, \sigma)$ be any signed graph. Then the Gallai signed graph $\mathcal{GL}(S)$ is balanced.

Proof. Let E_j^+ be the vertices set of the Gallai Signed Graph $\mathcal{GL}(S)$ each of which corresponds to a positive edge in signed graph S and E_j^- is the vertices set of the Gallai Signed Graph $\mathcal{GL}(S)$ each of which corresponds to a negative edge in Signed Graph S . From the characterization of the Gallai Signed Graph $\mathcal{GL}(S)$, the edges e_i and e_j of S are not of the identical Sign and hence as vertices of the Gallai Signed Graph $\mathcal{GL}(S)$, they cannot lie in the same part of partition $\{E_j^+, E_j^-\}$. On the other hand, if the edge $e_i e_j$ is any positive edge of the Gallai Signed Graph $\mathcal{GL}(S)$ then, by the characterization of Gallai Signed Graph $\mathcal{GL}(S)$, the edges e_i and e_j of the signed graph S are of the same sign and hence as vertices of $\mathcal{GL}(S)$ they must both lie in exactly one of the parts of the sub-sets $\{E_j^+, E_j^-\}$ of the set of vertex of the Gallai Signed Graph $\mathcal{GL}(S)$. Thus, every negative edge of the Gallai signed graph $\mathcal{GL}(S)$ has its ends in different parts of this partition whereas no positive edge of the Gallai signed graph $\mathcal{GL}(S)$ has this property. Hence, by Theorem 1, the Gallai signed graph $\mathcal{GL}(S)$ is balanced.

For any +ve integer k , the k^{th} repeated Gallai Signed Graph $\mathcal{GL}^k(S)$ of a signed graph S is well-defined as below:

$$\mathcal{GL}^0(S) = S, \mathcal{GL}^k(S) = \mathcal{GL}(\mathcal{GL}^{k-1}(S)).$$

Corollary 4. For any Signed Graph $S = (G, \sigma)$ and for any +ve integer k , k^{th} iterated Gallai Signed Graph $\mathcal{GL}^k(S)$ is balanced.

Theorem 5. Let $S_1 = (G_1, \sigma_1)$, $S_2 = (G_2, \sigma_2)$ with $G_1 \cong G_2$. Then $\mathcal{GL}(S_1) \sim \mathcal{GL}(S_2)$.

Proof. Let S_1 and S_2 be any signed graphs with their underlying graphs are isomorphic. Then $\mathcal{GL}(S_1)$ and $\mathcal{GL}(S_2)$ are balanced and they are switching equivalent, by Theorem 2.

In [6], the author characterize the graphs for which $\mathcal{GL}(G) \cong G$.

Theorem 6. Let $G = (V, E)$ be any Graph, Gallai Graph $\mathcal{GL}(G)$ is isomorphic to G if and only if, $G \cong C_n$, where $n \geq 4$.

In view of the above result, we now characterize the Signed Graphs for which Gallai Signed Graph $\mathcal{GL}(S)$ and S are switching equivalent.

Theorem 7. For any signed graph $S = (G, \sigma)$, the Gallai signed graph $\mathcal{GL}(S)$ and S are switching equivalent if and only if, S is Balanced and G is isomorphic to C_n , where $n \geq 4$.

Proof. Suppose that the Signed Graph S is Balanced and G is isomorphic to C_n , where $n \geq 4$. Then, since $\mathcal{GL}(S)$ as Theorem 3. Consider a signed graph $S = (G, \sigma)$ with underlying graph as a cycle with n vertices, where $n \geq 4$. Then $S = (G, \sigma)$ and $\mathcal{GL}(S)$ are balanced and hence they are switching equivalent, from the Theorem 2. Conversely, suppose that the Gallai signed graph $\mathcal{GL}(S)$ and S are switching equivalent. Then, G and $\mathcal{GL}(G)$ are isomorphic. Now G is a cycle with n vertices, where $n \geq 4$. Consider a signed graph $S = (G, \sigma)$ with $G = C_n$, where $n \geq 4$ then $\mathcal{GL}(S)$ is balanced. If S is negative (i.e, the product of the signs of each cycle in S is negative), then $S = (G, \sigma)$ and $\mathcal{GL}(S)$ are not switching equivalent, which is a contradiction and hence $S = (G, \sigma)$ and $\mathcal{GL}(S)$ are switching equivalent, by the hypothesis. Therefore, S is positive.

A. Characterization of Gallai Signed Graphs

In this subdivision, we presented the structural characterization of Gallai Signed Graphs.

Theorem 8. A Signed Graph $S = (G, \sigma)$ is a Gallai Signed Graph, if and only if, S is Balanced Signed Graph and its underlying graph G is a Gallai Graph.

Proof. Let us assume that S is a Gallai Signed Graph. Then $S \cong \mathcal{GL}(S_1)$, where $S_1 = (G_1, \sigma)$ is for some signed graph. Therefore, S balanced, since S is a Gallai signed graph. $S \cong \mathcal{GL}(S_1) \Rightarrow G \cong \mathcal{GL}(G_1)$ and this shows that G is the Gallai graph.

Conversely, assume that $S = (G, \sigma)$ is Balanced and G is a Gallai Graph (i.e., $G \cong \mathcal{GL}(G_1)$, for some graph G_1). By the hypothesis, S is balanced. According to the Sampathkumar's result (Theorem 1), each edge $e = pq \in E(S)$, the sign of e is equal to the product of the marks of p and q . Hence $S \cong \mathcal{GL}(S_1)$. Therefore, S is a Gallai signed graph.

III. ANTI-GALLAI SIGNED GRAPH OF A SIGNED GRAPH

The anti-Gallai Signed Graph $\mathcal{AGL}(S) = (\mathcal{AGL}(G), \sigma')$ is a Signed Graph and $\mathcal{AGL}(G)$ is an anti-Gallai graph and sign of any $e_1 e_2 \in E(\mathcal{AGL}(S))$ is the multiplication of the signs of the corresponding edges in $S = (G, \sigma)$. For some signed graphs S_i such that $\mathcal{AGL}(S_i) \cong \Sigma$ is the anti-Gallai signed graph.

Signed graphs can be classified into two types namely, balanced and unbalanced signed graphs. The signed graph $S = (G, \sigma)$ is balanced or unbalanced, we have $\mathcal{AGL}(S)$ is always balanced in either of the cases.

Theorem 9. Let $S = (G, \sigma)$ be any Signed Graph. Then the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ is Balanced.

Proof. Assume E_j^+ as the vertex set of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ each of which corresponds to a positive edge in signed graph S and E_j^- be the set of vertices of the anti-Gallai signed graph $\mathcal{AGL}(S)$ each of which corresponds to a negative edge in Signed Graph S . By the definition of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$, Edges e_i and e_j are not of the similar Sign and hence as vertices of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$, they cannot lie in the same part of partition $\{E_i^+, E_j^-\}$. On the other hand, if the edge $e_i e_j$ is any positive edge of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ then, as per the characterization of Anti-Gallai Signed Graph $\mathcal{AGL}(S)$, the edges e_i and e_j of the Signed Graph S are of the same sign and hence as vertices of $\mathcal{AGL}(S)$ they must both lie in exactly one of the parts of the group $\{E_i^+, E_j^-\}$ of the vertex set of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$. Thus, every negative edge of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ has its ends in different parts of this partition whereas no +ve edge of the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ has this property. Hence, by Theorem 1, the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ is Balanced.

Given, for any +ve integer k , the k^{th} iterated Anti-Gallai Signed Graph $\mathcal{AGL}^k(S)$ of a signed graph S is characterized as follows:

$$\mathcal{AGL}^0(S) = S, \mathcal{AGL}^k(S) = \mathcal{AGL}(\mathcal{AGL}^{k-1}(S)).$$

Corollary 10. In any Signed Graph $S = (G, \sigma)$ and for any +ve integer k , k^{th} iterated anti-Gallai signed graph $\mathcal{AGL}^k(S)$ is balanced.

Theorem 11. Let $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ with $G_1 \cong G_2$. Then $\mathcal{AGL}(S_1) \sim \mathcal{AGL}(S_2)$.

Proof. Let S_1 and S_2 be any signed graphs with their underlying graphs are isomorphic. Then $\mathcal{AGL}(S_1)$ and $\mathcal{AGL}(S_2)$ are balanced and they are switching equivalent, by Theorem 2.

In [6], the author characterize the graphs for which $\mathcal{AGL}(G) \cong G$.

Theorem 12. Let $G = (V, E)$ be any graph, anti-Gallai graph $\mathcal{AGL}(G)$ is isomorphic to G if, and only if, $G \cong K_3$.

Considering to the above result, we now characterize the Signed Graphs for which Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ and S are switching equivalent.

Theorem 13. For any Signed Graph $S = (G, \sigma)$, the Anti-Gallai Signed Graph $\mathcal{AGL}(S)$ and S are switching equivalent, if and only if S is Balanced and G is isomorphic to K_3 .

Proof. Presume that the Signed Graph S is Balanced and G is isomorphic to K_3 . Then, since $\mathcal{AGL}(S)$ as Theorem 9. Consider a Signed Graph $S = (G, \sigma)$ with underlying graph as a cycle with three vertices. Then $S = (G, \sigma)$, and $\mathcal{AGL}(S)$ are balanced and hence they are switching equivalent, from the Theorem 2.

Conversely, suppose that the anti-Gallai signed graph $\mathcal{AGL}(S)$ and S are switching equivalent. Then, G and $\mathcal{AGL}(G)$ are isomorphic. Now G is a cycle with 3 vertices. Consider a signed graph $S = (G, \sigma)$ with $G = C_3$ then $\mathcal{AGL}(S)$ is balanced. If S is negative (i.e, the product of the signs of each cycle in S is negative), then $S = (G, \sigma)$ and $\mathcal{AGL}(S)$ are not switching equivalent, which is a contradiction and hence $S = (G, \sigma)$ and $\mathcal{AGL}(S)$ are switching equivalent, by the hypothesis. Therefore, S is positive.

A. Characterization of Anti-Gallai Signed Graphs

In this sub-section, we presented the structural characterization of Anti-Gallai Signed Graphs.

Theorem 14. A Signed Graph $S = (G, \sigma)$ is an Anti-Gallai Signed Graph, if and only if, S is Balanced Signed Graph and its principal Graph G is an Anti-Gallai Graph.

Proof. Let us assume that S is an anti-Gallai signed graph. Then $S \cong \mathcal{AGL}(S_1)$, where $S_1 = (G_1, \sigma)$ is for some signed graph. Therefore, S balanced, since S is an anti-Gallai signed graph. $S \cong \mathcal{AGL}(S_1) \Rightarrow G \cong \mathcal{AGL}(G_1)$ and this shows that G is an anti-Gallai graph

Conversely, presume that $S = (G, \sigma)$ is Balanced and G is an Anti-Gallai Graph (i.e., $G \cong \mathcal{AGL}(G_1)$, for some graph G_1). By the hypothesis, S is balanced. According to the Sampathkumar's result (Theorem 1), each edge $e = pq \in E(S)$, the sign of e is equal to the product of the marks of p and q . Hence $S \cong \mathcal{AGL}(S_1)$. Therefore, S is an anti-Gallai signed graph.

In [7], the authors characterize graphs for which $\mathcal{GL}(G) \cong \mathcal{AGL}(G)$.

Theorem 15. For any Graph $G = (V, E)$, the Graphs $\mathcal{GL}(G)$ and $\mathcal{AGL}(G)$ are isomorphic, if and only if, G is $nK_3 \cup nK_{1,3}$.

Considering the above result, we now characterize the Signed Graphs for which Gallai Signed Graphs and Anti-Gallai Signed Graphs are switching equivalent.

Theorem 16. For any Signed Graph $S = (G, \sigma)$, $\mathcal{GL}(S)$ and $\mathcal{AGL}(S)$ are switching equivalent if, and only if, G is $nK_3 \cup nK_{1,3}$.

Proof. Now S is any signed graph with underlying graph G is $nK_3 \cup nK_{1,3}$, then $\mathcal{GL}(S)$ and $\mathcal{AGL}(S)$ are balanced. Hence, by Theorem 2, $\mathcal{GL}(S)$ and $\mathcal{AGL}(S)$ are switching equivalent.

Conversely, suppose that $\mathcal{GL}(S)$ and $\mathcal{AGL}(S)$ are switching equivalent. Then $\mathcal{GL}(G)$ and $\mathcal{AGL}(G)$ are isomorphic. By Theorem 15, G is isomorphic to $nK_3 \cup nK_{1,3}$.

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