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A New Method to Compute the Determinantal Polynomial Coefficients of a Star Graph

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***Abstract*—Considering this article, we have calculated the coefficients of characteristic (determinantal) polynomial for a Star graph by taking the vertex number and using the eigenvalues of the Graph's matrix.**
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I. INTRODUCTION

For all phrases and notes of graph theory, readers should refer to [35]. Here we consider simple and non-loop graphs including the restricted number of vertices and edges.
Laplacian coefficients of Characteristic polynomial of a Star Graph
In studying the chemical characteristics of molecules, coefficients of a characteristic polynomial are helpful.
The characteristic polynomial of a graph can be evaluated using a variety of techniques ([1] [30]). In [30], it is asserted that the Faddeev-Le Verrier- Frame approach ([14] [17]) is the most effective technique for calculating the characteristic polynomial.
Balasubramanian ([18][26]) frequently used the computerized version of the following variation of this technique.
The coefficients of the characteristic polynomial $p_{g(\lambda)} = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n$ of graph G are directly related to its structure.
For illustration, the number of edges of G is $-b_2$ and the number of triangles of G is $-b_3/2$. In [38] the following theorem determines the coefficients of the characteristic polynomial of matrix of a graph.
Theorem1.*[38]Let* $tp_{g(\lambda)} = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n$ be the characteristic polynomial of graph G, the sets E_i, $1 \le i \le n$ consisting of the elementary sub- graphs of G, with i vertices and $r(\wedge)$ and $s(\wedge)$ the rank and co-rank of E_i, respectively. We then have.

$$(-1)^i a_i = \sum_{\wedge \in E_i} (-1)^{r(\wedge)} 2^{s(\wedge)}$$

Using the trace of the Laplacian matrix, Ivailo M. Mladenov et al. have provided a very elegant approach to identify the coefficients of a characteristic polynomial. Samuel Jurkiewicz et al. has determined the Laplacian coefficients of a characteristic polynomial in [32].
We provide a new technique to discover Laplacian coefficients of characteristic polynomial using the trace of a Laplacian matrix of Star graph G, based on the afore mentioned theorem, Faddeev-LeVerrier algorithm, and [38].

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Proposition 2.Let G be a Star graph then,

$tr(L^l) = n^l + (n-2)$.

Proof. Here, the proof is by induction,

$$tr(L) = \sum_{i=1}^{n} \lambda_i$$
$$= n + (n-2)$$

we get

$$tr(L^2) = n^2 + (n-2)$$

and

$$tr(L^3) = n^3 + (n-2)$$

For an integer y,

$$tr(L^y) = n^y + (n-2)$$
$$tr(L^{y+1}) = n^y + (n-2)$$

Hence by induction tr $(L^l) = n^l + (n-2)$.
Example:
If we put l= 1, 2, 3, 4 in the above expression we have the following.

$\boldsymbol{tr(L)} = n+(n-2) = 2n-2$ $\boldsymbol{tr(L^2)} = n^2 + n-2$

$\boldsymbol{tr(L^3)} = n^3 + n-2$

$\boldsymbol{tr(L^4)} = n^4 + n-2$

Coefficients b_1, b_2, b_3 *and* b_4 of characteristic equation of a Laplacian matrix of a Star graph G, are calculated as follows:

$b_1 = -tr(L) = -n + (n - 2) = -(2n - 2)$

$b_2 = \frac{-1}{2} tr(B_1 L)$ $where B_1 = L + b_1 \mathbf{I}$

$= \frac{-1}{2} tr(L^2 - (2n - 2)L)$

$= \frac{-1}{2} (tr(L^2) - (2n - 2)tr(L))$

$= \frac{-1}{2} (n^2 + n - 2 - (n^2 - n)(n^2 - n))$

$= \frac{-1}{2} (-3n^2 + 9n - 6)$.

$$b_3 = \frac{-1}{3}\ tr(\ B_2\ L\ ) \qquad where \quad B_2 = B_1 L + b_2 \mathbf{I}$$
$$= \frac{-1}{3} tr\big((L + b_1 I)L^2 + b_2 L\big)$$

$= \frac{-1}{3} tr\big((L + b_1 I)L^2 + b_2 L\big)$

$= \frac{-1}{3} tr(L^3 + b_1 L^2 + b_2 L)$

$= \dfrac{-1}{3}\ (tr(L^3) + -(2n - 2)\ \ tr(L^2) + \dfrac{-1}{2}(-3n^2 + 9n - 6)tr(L))$

$b_4 = \frac{-1}{3}\ \ (2n^3 - 12n^2 + 22n - 12\ )$

$b_4 = \frac{-1}{4}tr(B_3L)$ where $B_3L + b_3I$

r

$= \frac{-1}{4}$tr $((B_1L + b_2)L^2 + b_3L)$

$= \ \frac{-1}{4}tr((L + b_1)L^3 + b_2L^2 + b_3L)$

$= \frac{-1}{4}tr(L^4 + b_1L^3 + b_2L^2 + b_3L)$

$= \frac{-1}{4}tr(tr(L^4) + b_1tr(L^3) + b_2tr(L^2) + b_3tr(L))$

$= \frac{-1}{24}(-5n^4 + 50n^3 - 175n^2 + 250n - 120)$

Theorem 3. The characteristic equation of a laplacian matrix of a star graph G is
Ψ (G ,t)=$t^n + b_1t^{n-1} + \cdots + b_{n-1}t + b_n$ and $b_0 = 1$ then ,

$$b_h = \frac{-1}{h}\sum_{r=0}^{h-1} b_r tr(L^{h-r})$$

Where b_h is the coefficients of the characteristic polynomial and $h \neq 0$.
Proof: here $b_1 = -tr(L)$

$$b_2 = \frac{-1}{2}tr(B_1L)$$

$= \frac{-1}{2}tr((\mathrm{L}+c)L)$

$= \frac{-1}{2}\{\mathrm{tr}(L^2) + \ b_1tr(\mathrm{L})$

$= \frac{-1}{2}\{b_0tr(L^2) + \ b_1tr(L)\}$.

For An integer y,

$$b_y = \frac{-1}{y}\sum_{r=0}^{y-1} b_r tr(\ L^{y-r}).$$

We have,

$$b_{y+1} = \frac{-1}{y+1}tr\left(B_yL\right)$$

$= \frac{-1}{y+1}tr\left(\left(B_{y-1}L + b_y\right)L\right)$

$= \frac{-1}{y+1}\mathrm{tr}((LB_{y-2} + b_{y-1})L^2 + b_y\mathrm{L})$

$= \frac{-1}{y+1}\left(tr\left(L^3B_{y-2}\right) + b_{y-1}\mathrm{tr}(L^2) + b_ytr(L)\right)$

$= \frac{-1}{y+1}(\mathrm{tr}\ (L^4B_{y-3}) + \mathrm{tr}(L^3)\ b_{y-1} + tr(L^2)b_y + tr(L))$

……

$$b_{k+1} = \frac{-1}{y+1}\{b_0tr(L^{y+1}) + b_1tr(L^y) + b_2tr(L^{y-1}) + \cdots + b_ytr(L)\}$$

i.e . , $b_{y+1} = \frac{-1}{y+1} \sum_{r=0}^{y} b_r tr(L^{y+1-r})$

Hence by induction,

$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r tr(L^{h-r})$ *where* h$\neq 0$

Theorem 4. Let *G* be a Star graph the laplacian characteristics polynomial of *G* is

$\chi(G, t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$ *with* $b_0 = 1$ then

$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r \left(n^{h-r} + (n-2)\right)$ *where* h$\neq 0$

Proof: by proposition 2, we have

tr $(L^{h-r}) = n^{h-r} + (n-2)$.

By theorem 3, we get

$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r \left(n^{h-r} + (n-2)\right)$ *where* h$\neq 0$

Theorem 5. Let G be a Star graph, the laplacian characteristics polynomial of G is
$\chi(G, t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$ *with* $b_0 = 1$ and λ_1 is the Laplacian eigenvalue of a star graph *G* then

$$b_\varsigma = \frac{-1}{\varsigma} \sum_{r=0}^{\varsigma-1} b_r((\lambda_1 - 2) + \lambda_1{}^{\varsigma-r})$$

Proof. Since, $\lambda_1 = n$ and hence by theorem 4 ,

$b_\varsigma = \frac{-1}{\varsigma} \sum_{r=0}^{\varsigma-1} b_r((\lambda_1 - 2) + \lambda_1{}^{\varsigma-r})$.

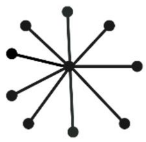
II. Conclusion

In this article, we have found the first two techniques for finding the coefficients of the Star graphs Laplacian matrix distinctive polynomial

- Using the table below indicated several vertices in a Star graph G
- Using Star graph G' s eigen values

TABLE I

Number of vertices (n) of a Star graph	Star Graph	Coefficients of characteristic polynomial
5		$b_1 = -8$, $b_2 = 18$, $b_3 = -16, b_4 = 5$
6		$b_1 = -10, b_2 = 30, b_3 = -40, b_4 = 25, b_5 = -6$
7		$b_1 = -12, b_2 = 45, b_3 = -80. b_4 = 75, b_5 = -36, b_6 = -7$
8		$b_1 = -14, b_2 = 63, b_3 = -140, b_4 = 175,$ $b_5 = -126, b_6 = 49, b_7 = -8$

9		$b_1 = -16, b_2 = 84, b_3 = -224, b_4 = 350,\ b_5 = -336,\ b_6 = 196, b_7 = 64,\ b_8 = 9$
10		$b_1 = -18, b_2 = 108, b_3 = -336,\ b_4 = 630, b_5 = -756, b_6 = 588, b_7 = -288, b_8 = 81, b_9 = -10$

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