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A Note on Laplacian Coefficients of the Characteristic Polynomial of a Complete Graph

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***Abstract*—We have discovered the distinctive polynomial of the complete graph's Laplacian matrix with this article in mind. The trace of the Laplacian matrix and the total number of vertices in the complete graph were used to determine the coefficients of the characteristic polynomial. Additionally, we have demonstrated that Laplacian eigenvalues can be used to get coefficients for identical characteristic polynomials. Can be used to get coefficients for identical characteristic polynomials.**

***Index Terms*— Graphs, complete graph, Laplacian Matrix.**

I. INTRODUCTION

For all phrases and notes of graph theory, readers should refer to F. Harary [12]. Here we consider simple and non-loop graphs including the restricted number of vertices and edges In studying the chemical characteristics of molecules, coefficients of a characteristic polynomial are helpful.
Various techniques can be employed to assess the characteristic polynomial of a graph. ([1] to [5]), it is asserted that the Faddeev LeVerrier- Frame approach ([6] , [7]) is the best method for determining the characteristic polynomial. Balasubramanian ([8], [9]) frequently used the computerized version of the following variation of this technique.
The characteristic polynomial of a graph G is $p_g(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n$.

Coefficients of this polynomial are directly related to its structure.
As an example, consider the fact that G has $-b_2$ the number of edges of G and the number of triangles of G is $^{-b_3}/_2$ mentioned in N. Biggs [10] following theorem determines the coefficients.

Theorem1.*[10]* Let $p_g(\lambda) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \cdots + b_n$ be the characteristic polynomial of graph G, the sets E_i, $1 \le i \le n$ consisting of the elementary sub graphs of G, with i vertices and r(∧) and s(∧) the rank and co-rank of E_i, respectively. We then have;

$$(-1)^i a_i = \sum_{\wedge \in E_i} (-1)^{r(\wedge)}\, 2^{s(\wedge)}$$

Using the trace of the Laplacian matrix, Ivailo M. Mladenov et al. have provided a very elegant approach to identify the coefficients of a characteristic polynomial. Samuel Jurkiewicz et al. has determined the Laplacian coefficients of a characteristic polynomial in [11].

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We provide a new technique to discover Laplacian coefficients of a characteristic polynomial using the trace of a Laplacian matrix of complete graph *G*, based on the aforementioned theorem, Faddeev-LeVerrier algorithm, and N. Biggs [10].

Proposition 2. Let G be a complete graph then,

$$tr(L^l) = n^l(n-1).$$

Proof. Here, the proof is by induction,

$$tr(L) = \sum_{i=1}^{n} \lambda_i$$

$$= n(n-1)$$

We get

$$tr(L^2) = n^2(n-1)$$

and

$$tr(L^3) = n^3(n-1)$$

For an integer y,

$$tr(L^y) = n^y(n-1)$$

$$tr(L^y) = \sum_{i=1}^{y+1} \lambda_i$$

$$tr(L^{y+1}) = n^y(n-1)$$

Hence by induction, $tr(L^l) = n^l(n-1)$

Example:

If we put *l* = 1, 2, 3, 4 in the above expression we have the following.

$\boldsymbol{tr(L)} = n(n-1) = n^2 - n$ $\boldsymbol{tr(L^2)} = n^3 - n^2$
$\boldsymbol{tr(L^3)} = n^4 - n^3$
$\boldsymbol{tr(L^4)} = n^5 - n^4$

Coefficients b_1, b_2, b_3 *and* b_4 of equation are calculated as follows:

$$b_1 = -\,tr\,(L) = -n(n-1)$$

$$b_2 = \frac{-1}{2} tr(B_1 L) \; where B_1 = L + b_1 \mathbf{I}$$

$$= \frac{-1}{2} tr(L^2 + (-n^2 + n)L)$$

$$= \frac{-1}{2}\left(tr(L^2) - (n^2 - n)tr(L)\right)$$

$$= \frac{-1}{2}\left(n^2(n-1) - (n^2 - n)(n^2 - n)\right)$$

$$= \frac{-1}{2}(-n^4 + 3n^3 - 2n^2)$$

$$b_3 = \frac{-1}{3} tr(B_2 L) where B_2 = B_1 L + b_2 \mathbf{I}$$

$$= \frac{-1}{3} tr\big((L + b_1 I)L^2 + b_2 L\big)$$

$$= \frac{-1}{3} tr\big((L + b_1 I)L^2 + b_2 L\big)$$

$$= \frac{-1}{3} tr(L^3 + b_1 L^2 + b_2 L)$$

$$= \frac{-1}{3} \left(tr(L^3) + (-n^2 + n)tr(L^2) + \frac{-1}{2}(-n^4 + 3n^3 - 2n^2)tr(L)\right)$$

$$= \frac{-1}{6} \left(n^6 - 6n^5 + 11n^4 - 6n^3\right)$$

$b_4 = \frac{-1}{4} tr(B_3 L)$ where $B_3 = B_2 L + b_2 \mathbf{I}$

$$= \frac{-1}{4} tr\left((B_1 L + b_2)L^2 + b_3 L\right)$$

$$= \frac{-1}{4} tr\left((L + b_1)L^3 + b_2 L^2 + b_3 L\right)$$

$$= \frac{-1}{4} tr(L^4 + b_1 L^3 + b_2 L^2 + b_3 L)$$

$$= \frac{-1}{4} \left(tr(L^4) + b_1 tr(L^3) + b_2 tr(L^2) + b_3 tr(L)\right)$$

$$= \frac{-1}{24} (n^8 - 10n^7 + 35n^6 + 24n^4 - 50n^5).$$

Theorem 3: The laplacian matrix characteristic equation of a complete graph G is $\Psi(G, t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$ and $b_0 = 1$ then

$$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r tr(L^{h-r})$$

Where b_h are the coefficients of the polynomial and $h \neq 0$.

Proof: Here $b_1 = -tr(L)$

$$b_2 = \frac{-1}{2} tr(B_1 L)$$

$$= \frac{-1}{2} tr((L + c)L)$$

$$= \frac{-1}{2} \{tr(L^2) + b_1 tr(L)$$

$$= \frac{-1}{2} \{b_0 tr(L^2) + b_1 tr(L)\}.$$

For an integer y,

$$b_y = \frac{-1}{y} \sum_{r=0}^{y-1} b_r tr(L^{y-r}).$$

We have ,

$$b_{y+1} = \frac{-1}{y+1} tr(B_y L)$$

$$= \frac{-1}{y+1} tr\Big(\big(B_{y-1} L + b_y\big)L\Big)$$

$$= \frac{-1}{y+1} tr\left((LB_{y-2} + b_{y-1})L^2 + b_y L\right)$$

$$= \frac{-1}{y+1} \left(tr\big(L^3 B_{y-2}\big) + b_{y-1} tr(L^2) + b_y tr(L)\right)$$

$$= \frac{-1}{y+1} \left(tr(L^4 B_{y-3}) + tr(L^3) b_{y-1} + tr(L^2) b_y + tr(L)\right)$$

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$$b_{k+1} = \frac{-1}{y+1} \{b_0 tr(L^{y+1}) + b_1 tr(L^y) + b_2 tr(L^{y-1}) + \cdots + b_y tr(L)\}$$

i.e . , $b_{y+1} = \frac{-1}{y+1} \sum_{r=0}^{y} b_r tr(L^{y+1-r})$

Hence by induction,

$$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r tr(L^{h-r}) \qquad where\ h \neq 0$$

Corollary 4: The Laplacian matrix equation of a complete graph *G* is

$\chi(G, t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$ $with\ b_0 = 1$ then

$$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r \left(n^{h-r}(n-1)\right) \ where\ h \neq 0$$

Proof: By proposition 2, we have

$$tr(L^{h-r}) = n^{h-r}(n-1).$$

By theorem 3, we get

$$b_h = \frac{-1}{h} \sum_{r=0}^{h-1} b_r \left(n^{h-r}(n-1)\right) \ where\ h \neq 0$$

Corollary 5: The laplacian characteristic equation of a complete graph G is $\chi(G, t) = t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n$ $with\ b_0 = 1$ and λ_1 is the maximum eigenvalue of a Laplacian matrix then

$$b_\varsigma = \frac{-1}{\varsigma} \sum_{r=0}^{\varsigma-1} b_r \left((1-\lambda_1){\lambda_1}^{\varsigma-r}\right)$$

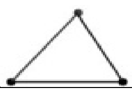
Proof: Since, $\lambda_1 = n$ and hence by Corollary 4.

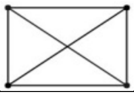
$$b_\varsigma = \frac{-1}{\varsigma} \sum_{r=0}^{\varsigma-1} b_r \left((1-\lambda_1){\lambda_1}^{\varsigma-r}\right)$$

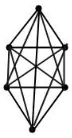
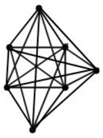
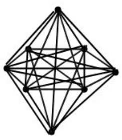
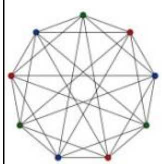
II. CONCLUSION

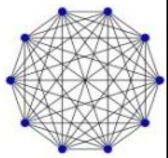
In this article, have found the first two techniques for finding the coefficients of graph's Laplacian matrix distinctive polynomial

i. Using number of vertices as shown in the following table.
ii. Using the complete graph G' $s\ eigen\ values$

Number of vertices (n) of a complete graph	Graph	Coefficients of characteristic polynomial
3		b_1 = -6, b_2 = 9, b_3 = - 3

4		b_1 = -12, b_2 = 48, b_3 = -64, b_4 = 0
5		b_1 = -20, b_2 = 150, b_3 = -500, b_4 = 625, b_5 = 0

Number of vertices (n) of a complete graph	Graph	Coefficients of characteristic polynomial
6		b_1 = -30, b_2 = 360, b_3 = -2160, b_4 = 6480, b_5 = -7776, b_6 = 0
7		b_1 = -42, b_2 = 735, b_3 = -6860, b_4 = 36015, b_5 = -100842, b_6 = 117649, b_7 = 0
8		b_1 = -56, b_2 = 1344, b_3 = -17920, b_4 = 143360, b_5 = -688128 b_6 = 1835008, b_7 = -2097152, b_8 = 0
9		b_1 = -72, b_2 = 2268, b_3 = -40824, b_4 = 459270, b_5 = -3306744, b_6 = 14880348, b_7 = -38263752, b_8 = 43046721, b_9 = 0

10		b_1 = -90, b_2 = 3600, b_3 = -84000, b_4 = 1260000, b_5 = -12600000, b_6 = 84000000, b_7 = -360000000, b_8 = 900000000, b_9 = -1000000000, b_{10} = 0

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