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Design and Analysis of Derivative-Free Iterative Approaches to Solving Nonlinear Equations

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***Abstract*— In this manuscript, we propose a new method to solve nonlinear equations without the need for higher-order derivatives. In particular, we propose a fourth-order, derivative-free iterative scheme for approximating the roots of nonlinear functions. The convergence analysis of the proposed method is carried out in two steps: first, to show its derivative-free nature, and second, to include a proper weight function. For comparing the efficiency of the method proposed, we conduct comparative analyses with some popular methods for a group of nonlinear equations. The results indicate that our method is more efficient than other methods in all the cases. Numerical experiments also validate the efficiency and accuracy of the method, and show better performance in CPU time, residual error, reduction of error per iteration, and number of iterations to converge.**

***Keywords*— Order of Convergence, Non-Linear Equations, Derivative Free Iterative Method.**

I. Introduction

In recent years, numerous studies have explored the idea of eliminating derivatives from iterative functions to keep from computing new operations, which include the first and second derivatives. The goal is to compute iterates solely using the original function that defines the problem, while preserving the convergence order. In this context, the term "derivative-free" [3, 12, 14] is commonly used in the literature on nonlinear equations, typically referring to methods that bypass the derivative. Finding efficient techniques for resolving solutions to nonlinear equations remains a crucial research area in the field of numerical analysis, as such equations arise in various scientific and technological disciplines.

In this study, we focus on solving scalar nonlinear problems in the format of: $F(r) = 0$, where $F: D \subset R \to R$, D is an open set. Over the past decades, iterative methods have gained widespread use for solving nonlinear problems [2], generating sequences of points that progressively approach the root, ultimately yielding an approximation with the desired precision. Extensive literature exists on methods for approximating the solutions of nonlinear equations. Among the most recognized methods is Newton's iteration method [1, 8], that is derived from" the iterative expression:

$$r_{k+1} = r_k - \frac{F(r_k)}{F'(r_k)}$$

Given an initial guess that is not the exact root, Newton's method exhibits quadratic convergence [15,16].

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However, in many cases, the derivatives of nonlinear equations are either difficult to compute or lack an explicit analytical expression. To address this, Steffensen [6] proposed approximating the derivative in Newton's method as:

$$F'(r_k) \approx \frac{F(r_k + F(r_k)) - F(r_k)}{F(r_k)}$$

Substituting this approximation into Newton's iterative formula yields Steffensen's method [6]:

$$r_{k+1} = r_k - \frac{F(r_k)^2}{F(r_k + F(r_k)) - F(r_k)}$$

which also achieves quadratic convergence. The incorporation of weight functions in iterative schemes allows for the development of methods that improve Convergence rate without significantly improving the number of evaluation of functions. In the case of this paper, we utilize these techniques to design methods that enhance convergence order [4, 5, 10, 11] while minimizing computational costs. The composition of the paper is organized as follows: Section 2 outlines the development of an innovative iterative technique, along with a proof in terms of its convergence order. Section 3 analyzes the numerical results, including comparisons with existing methods to validate the theoretical findings. Finally, Section 4 concludes the paper by summarizing the results and suggesting potential future directions.

II. Construction and Convergence Analysis of Proposed Method

Considering Newton's Modified Method [7], which is expressed as:

$$y_k = r_k - \frac{F\left(r_k - \frac{F(r_k)}{F'(r_k)}\right) + F(r_k)}{F'(r_k)} \tag{2.1}$$

In this equation, $F'(r_k)$ represents the derivative of the function $F(r)$. By approximating the derivative as

$$F'(r) \approx \frac{F(r+h) - F(r)}{h}$$

equation (2.1) reduces to

$$y_k = r_k - \left(\frac{F\left(r_k - \frac{F(r_k)^2}{F(r_k + F(r_k)) - F(r_k)}\right) + F(r_k)}{F(r_k + F(r_k)) - F(r_k)}\right) F'(r_k) \tag{2.2}$$

The proposed method is constructed by taking the first step as equation (2.2). In the second step, a weight function is introduced, and the method PM is defined as follows:

$$y_k = r_k - \left(\frac{F\left(r_k - \frac{F(r_k)^2}{F(r_k + F(r_k)) - F(r_k)}\right) + F(r_k)}{F(r_k + F(r_k)) - F(r_k)}\right) F'(r_k)$$

$$r_{k+1} = r_k - H(\mu_k) \left(\frac{F\left(r_k - \frac{F(r_k)^2}{F(r_k + F(r_k)) - F(r_k)}\right) + F(r_k)}{F(r_k + F(r_k)) - F(r_k)}\right) F'(r_k) \tag{2.3}$$

in which the weight function $H(\mu_k): R \to R$ is defined with its origin at 0, and $\mu_k = \frac{F(y_k)}{F(r_k)}$. The weight function $H(\mu_k)$ is subjected to specific constraints that ensure the convergence of the iterative scheme under appropriate conditions. These constraints and their implications for convergence are discussed in detail in Theorem 2.1.

A. Convergence Analysis

The theorem presented here outlines the analysis of convergence for a class of proposed method (2.3) to determine the solutions of nonlinear equations.

Theorem 2.1: Let $F: D \subset R \to R$ be a nonlinear expression defined as part of a neighborhood D containing α, where $F(\alpha) = 0$. Assume that $F'(\alpha) \not\equiv 0$. If r_0 is an initial guess sufficiently near to the actual value that satisfies the equation, then the proposed method described in the equation (2.3) exhibits fourth-order convergence, assuming the weight function satisfies

$$H(0) = 1,\ H'(0) = 1.$$

Proof: Let $e_k = r_k - \alpha$ represent The deviation in the K^{th} repetition . Then, we can express the iteration as

$$r_k = e_k + \alpha \tag{2.4}$$

Substituting this value into the function, we have

$$\mathrm{F}(r_k) = \mathrm{F}(e_k + \alpha). \tag{2.5}$$

Now, applying Taylor expansion to $\mathrm{F}(r_k)$ around α, we get

$$\mathrm{F}(r_k) = \mathrm{v}(\alpha) + e_k\, \mathrm{F}'(\alpha) + \frac{e_k^2\, \mathrm{F}''(\alpha)}{2!} + \frac{e_k^3 \mathrm{F}'''(\alpha)}{3!} + \ldots. \tag{2.6}$$

Since α is the actual root, we know that F(α) = 0. Substituting this, the equation (2.6) simplifies to

$$\mathrm{F}(r_k) = e_k \mathrm{F}'(\alpha) + \frac{e_k^2\, \mathrm{F}''(\alpha)}{2!} + \frac{e_k^3\, \mathrm{F}'''(\alpha)}{3!} + \ldots\ldots. \tag{2.7}$$

Factoring out F′(α) from equation (2.7), we obtain

$$\mathrm{F}(r_k) = \mathrm{F}'(\alpha)\left(\mathrm{e_k} + \frac{\mathrm{e_k^2}}{2!}\frac{\mathrm{F}''(\alpha)}{\mathrm{F}'(\alpha)} + \frac{\mathrm{e_k^3}}{3!}\frac{\mathrm{F}'''(\alpha)}{\mathrm{F}'(\alpha)} + \ldots\right). \tag{2.8}$$

Let $c_i = \frac{\mathrm{F}^{(i+1)}(\alpha)}{(i+1)!\,\mathrm{F}'(\alpha)}$. Then, from equation (2.8), the value of $\Phi(r_k)$ becomes

$$\mathrm{F}(r_k) = \mathrm{F}'(\alpha)\left(e_k + c_1 e_k^2 + c_2 e_k^3 + \cdots\right). \tag{2.9}$$

Next, let $t_k = r_k + \mathrm{F}(r_k)$. Substituting the expression for $\mathrm{F}(r_k)$, we get

$$t_k = 2e_k + c_1 e_k^2 + c_2 e_k^3 + \mathrm{O}(e_k^4). \tag{2.10}$$

Now, applying the Taylor expansion to $\mathrm{F}(t_k)$ around α, we obtain

$$\mathrm{F}(t_k) = t_k + c_1 t_k^2 + c_2 t_k^3 + \mathrm{O}(t_k^4). \tag{2.11}$$

Substituting the expression for t_k, this becomes

$$\mathrm{F}(t_k) = 2e_k + 5c_1 e_k^2 + (4c_2^2 + 9c_2)e_k^3 + \mathrm{O}(e_k^4). \tag{2.12}$$

Next, computing the value of $r_k - \frac{(\mathrm{F}(r_k))^2}{\mathrm{F}(r_k + \mathrm{F}(r_k)) - \mathrm{F}(r_k)}$ from the first step of the proposed scheme (2.3), we denote this result by l_k.
Now,

$$l_k = r_k - \frac{(\mathrm{F}(r_k))^2}{\mathrm{F}(r_k + \mathrm{F}(r_k)) - \mathrm{F}(r_k)}$$

$$= 2c_1 e_k^2 + (-5c_1^2 + 6c_2)e_k^3 + (13c_1^3 - 2c_1 c_2 + 14c_1)e_k^4 + \mathrm{O}(e_k^5). \tag{2.13}$$

Now, applying the Taylor expansion to $\mathrm{F}(l_k)$ around α, we have

$$\mathrm{F}(l_k) = \mathrm{F}\left(\mathrm{r_k} - \frac{(\mathrm{F}(r_k))^2}{\mathrm{F}(r_k + \mathrm{F}(r_k)) - \mathrm{F}(r_k)}\right)$$

$$= 2c_1 e_k^2 + (-5c_1^2 + 6c_2)e_k^3 + (17c_1 c_3 - 2c_1 c_2 + 14c_3)e_k^4 + \mathrm{O}(e_k^5). \tag{2.14}$$

Next, we compute the values of $(\mathrm{F}(l_k) + \mathrm{F}(r_k))\cdot\mathrm{F}(r_k)$ and $\mathrm{F}(r_k + \mathrm{F}(r_k)) - \mathrm{F}(r_k)$, denoted as m_k and n_k, respectively:

$$m_k = e_k^2 + 4c_1 e_k^3 + (-2c_1^2 + 8c_2)e_k^4 + \mathrm{O}(e_k^5). \tag{2.15}$$

$$n_k = e_k + 4c_1 e_k^2 + (4c_1^2 + 8c_2)e_k^3 + (c_1^3 + 16c_1 c_2 + 16c_3)e_k^4 + \mathrm{O}(e_k^5). \tag{2.16}$$

Substituting the expressions from equations (2.15) and (2.16) into the first step of the proposed scheme (2.3), we get

$$y_k = r_k - \frac{e_k^2 + 4c_1e_k^3 + (-2c_1^2 + 8c_2)e_k^4 + O(e_k^5)}{e_k + 4c_1e_k^2 + (4c_1^2 + 8c_2)e_k^3 + (c_1^3 + 16c_1c_2 + 16c_3)e_k^4 + O(e_k^5)},$$

$$= r_k - \left(e_k - 6c_1^2e_k^3 + (35c_1^3 - 32c_1c_2)e_k^4 + O(e_k^5)\right). \qquad (2.17)$$

Subtracting α from both sides of equation (2.17), and letting $e'_k = y_k - \alpha$, we obtain

$$y_k - \alpha = r_k - \alpha - \left(e_k - 6c_1^2e_k^3 + (35c_1^3 - 32c_1c_2)e_k^4 + O(e_k^5)\right) \qquad (2.18)$$

Thus, (2.18) becomes

$$e'_k = e_k - \left(e_k - 6c_1^2e_k^3 + (35c_1^3 - 32c_1c_2)e_k^4 + O(e_k^5)\right).$$

$$= 6c_1^2e_k^3 + (35c_1^3 - 32c_1c_2)e_k^4 + O(e_k^5). \qquad (2.19)$$

Now, applying Taylor's expansion to $F(y_k)$ around α, we have:

$$F(y_k) = F(e'_k + \alpha),$$

$$= F'(\alpha)(e'_k + c_1e'^2_k + c_2e'^3_k) + O(e'^4_k).$$

$$= 6c21\, e_k^3 + (-35c_1^3 + 32c_1c_2)e_k^4 + O(e_k^5). \qquad (2.20)$$

Now, using the values of $F(y_k)$ and $F(r_k)$ from equations (2.20) and (2.9), we calculate the value of μ_k:

$$\mu_k = \frac{F(y_k)}{F(r_k)},$$

$$= \frac{6c_1^2e_k^3 + (-35c_1^3 + 32c_1c_2)e_k^4 + O(e_k^5)}{e_k + c_1e_k^2 + c_2e_k^3 + O(e_k^4)} \qquad (2.21)$$

Substituting the value of e'_k from equation (2.18) into equation (2.21), we get:

$$\mu_k = 6c_1e_k^2 + (-41c_1^3 + 32c_1c_2)e_k^3 + (194c_1^4 - 295c_1^2c_2 + 42c_2^2 + 72c_1c_3)e_k^4 + O(e_k^5). \qquad (2.22)$$

Therefore, the weighting factor $H(\mu_k)$ close to zero is expressed as:

$$H(\mu_k) = H(0) + H'(0)\mu_k + \frac{H''(0)}{2!}\mu_K^2 + O(\mu_K^3),$$

$$= H(0) + 6H'(0)e_k^2 + (-41c_1^3 + 32c_1c_2)H'(0)e_k^3 + [(194c_1^4 - 295c_1c_2 + 42c_2^2 + 72c_1c_3)H'(0) + 18c_1^4H''(0)]\, e_k^4 + O(e_k^5). \qquad (2.23)$$

Now, using the value of $H(\mu_K)$ and substituting equations (2.15) and (2.16) into the second step of the proposed scheme, we have:

$$r_{k+1} = r_k - H(\mu_k).\, F\left(\frac{\left(r_k - \frac{F(r_k)^2}{F(r_k + F(r_k)) - F(r_k)}\right) + F(r_k)}{F(r_k + F(r_k)) - F(r_k)}\right) F(r_K). \qquad (2.24)$$

Subtracting α from both sides of the above equation:

$$e_{k+1} = e_k - \left(H(0)e_k + (-6c_1^2H(0) + 6c_1^2H'(0)e_k^3 + (35c_1^3H(0) - 32c_1c_2H(0) - 41c_1^3H'(0) + 32c_1c_2H'(0))e_k^4 + O(e_k^5)\right).$$

$$= (1 - H(0))e_k + (6c_1^2H(0) - 6c_1^2H'(0)e_k^3 + (-35c_1^3H(0) + 32c_1c_2H(0) + 41c_1^3H'(0) + 32c_1c_2H'(0))e_k^4 + O(e_k^5). \qquad (2.25)$$

To demonstrate that the proposed method achieves fourth-order convergence in the equation scheme (2.3), the coefficients e_k the terms e_k and e_k^3 must be zero. From equation (2.25), we obtain the system outlined below:

$$\begin{cases} H(0) - 1 = 0, \\ H(0) - H'(0) = 0. \end{cases} \qquad (2.26)$$

Thus, the solution of the system is:

$$H(0) = 1, H'(0) = 1. \qquad (2.27)$$

Substituting these values into equation (2.25), the error equation of the proposed method can be written as:

$$e_{k+1} = 6c_1^5e_k^4 + (-5c_1^4 + 38c_1^2c_2 - 18c_1^4H'''(0))e_k^5 + O(e_k^6). \tag{2.28}$$

This completes the proof demonstration.

III. Numerical Experiments

This section presents a numerical comparison of the proposed fourth-order family method (2.3) with other established methods for solving nonlinear equations that share the same order of convergence. In each case, the families are developed using the method of weight functions.

The members of class (2.3) used are H_1, H_2, and H_3 which align with the weight functions $H_1(\mu_k) = 1 + \mu_k + \mu_k^2$, $H_2(\mu_k) = \frac{2\mu_k + 1}{\mu_k + 1}$ and $H_3(\mu_k) = \frac{1+2\mu_k}{1 + \mu_k + \mu_k^2}$ respectively.

Tables [1 – 3], shows the findings obtained using these methods. All computing has been carried out using Mathematica software version 11.1.1 and the stopping criteria $|r_{k+1} - r_k| < 10^{-200}$ are used. Furthermore, the rate of convergence in computation (ACOC) was approximated using the following algor

$$\text{ACOC} \approx \frac{\ln\left]\frac{r_{k+2}-r_{k+1}}{r_{k+1}-r_k}\right[}{\ln\left]\frac{r_{k+1}-r_k}{r_k-r_{k-1}}\right[}$$

The significance of the expression m(±n) is m × $10^{(\pm n)}$ evident throughout the entire table. In order to facilitate for the sake of comparison, we have employed the fourth-order method used in cordero et.al. [9] and denoted by AC as follows:

$$y_k = r_k - \frac{F(r_k)}{F[w_k, y_k]}$$

$$r_{k+1} = y_k - H(\mu_k)\frac{F(y_k)}{F[y_k, r_k]}$$

where $w_k = r_k + \beta F(r_k)$ the variable for the weight function is defined as follows $\mu_k = \frac{F(y_k)}{F(w_k)}$

Furthermore, we have implemented the fourth order as described by Chicharro et.al. [13], which is denoted as F C, as follows

$$y_k = r_k - \frac{F(r_k)}{F'(r_k)}$$

$$r_{k+1} = r_k - H(\mu_k)\frac{F(r_k)}{F'(r_k)}$$

$$\text{where } \mu_k = \frac{F(y_k)}{F(r_k)}$$

Example 3.1 Consider a nonlinear equation presented in the following manner

$F_1(r) = \sin(r) - r^2 + 1$

The actual root of the function is $\alpha \approx 1.409624$ and the initial guess is taken as 1.

This table provides a comparative evaluation of various methods (PM_1, PM_2, PM_3, AC_1, AC_2, FC_1, FC_2) grounded in numerical performance. All methods have ACOC = 4.0000, exhibit a comparable convergence order.

TABLE I: NUMERICAL OUTCOMES DERIVED FROM EXAMPLE 3.1 USING DIFFERENT METHODOLOGIES, FOR INSTANCE

Method	$\lvert r_{(3)} - r_{(2)}\rvert$	$\lvert r_{(4)} - r_{(3)}\rvert$	$\lvert r_{(5)} - r_{(4)}\rvert$	$F(r_{(5)})$	ACOC	CPU Timing
PM_1	2.8(−12)	1.2(−47)	4.5(−189)	2.2(−754)	4.0000	1.344
PM_2	1.2(−12)	5.4(−49)	1.7(−194)	4.8(−776)	4.0000	1.422
PM_3	8.1(−13)	8.6(−50)	1.1(−197)	7.7(−789)	4.0000	1.376
AC_1	2.4(−8)	3.3(−31)	1.1(-122)	5.1(−468)	4.0000	2.328
AC_2	3.9(−9)	1.5(−34)	4.3(−136)	6.3(−542)	4.0000	2.188
FC_1	9.2(−4)	6.3(−13)	1.4(−49)	1.0(−195)	4.0000	3.749
FC_2	4.0(−9)	4.5(−35)	7.3(−139)	1.3(−553)	4.0000	3.188

Example 3.2 Consider a nonlinear equation presented in the following manner

$$F_2(r) = e^{-r} + 2\sin(r) - r + 3.5$$

The actual root of the function is $\alpha \approx 3.273938$ and the initial guess is taken as 3.
These columns quantify the absolute differences observed between consecutive iterations of a numerical algorithm.
Reduced values signify enhanced accuracy and more rapid convergence. Error metrics ($|r_{(3)}-r_{(2)}|$, $|r_{(4)}-r_{(3)}|$, $|r_{(5)}-r_{(4)}|$), represented by these columns, quantify the absolute differences between successive iterations of a numerical algorithm. Lower values signify greater accuracy and more rapid convergence.

TABLE II: NUMERICAL OUTCOMES DERIVED FROM EXAMPLE 3.2 USING DIFFERENT METHODOLOGIES, FOR INSTANCE.

Method	$\lvert r_{(3)} - r_{(2)}\rvert$	$\lvert r_{(4)} - r_{(3)}\rvert$	$\lvert r_{(5)} - r_{(4)}\rvert$	$F(r_{(5)})$	ACOC	CPU Timing
PM_1	1.4(−26)	1.2(−107)	6.2(−432)	9.2(−1728)	4.0000	4.218
PM_2	1.0(−25)	3.5(−104)	3.9(−418)	1.8(−1673)	4.0000	4.188
PM_3	2.2(−25)	6.1(−103)	3.6(−413)	1.4(−1653)	4.0000	4.048
AC_1	6.6(−17)	4.5(−67)	9.6(-268)	6.6(−1070)	4.0000	5.125
AC_2	5.8(−17)	2.5(−67)	1.0(−268)	7.5(−1074)	4.0000	5.983
FC_1	5.3(−24)	4.8(−96)	3.3(−384)	2.3(−1536)	4.0000	4.390
FC_1	4.8(−24)	3.0(−96)	4.5(−385)	6.8(−1540)	4.0000	4.423

Example 3.3 An unwanted RF breakdown [17], which can occur in high-power microwave devices operating in a vacuum, is referred to as the multifactor effect. This phenomenon, for example, can be observed within a parallelplate waveguide. The presence of an electric field, along with the potential difference, induces electron movement between the two plates. Specifically, the path of an electron within the gap of air between two parallel plates can be described in the following manner:

$$y(t) = y_0 + \left(v_0 + e\frac{E_0}{m\omega}\sin(\omega t_0 + \alpha)\right)(t - t_0) + e\frac{E_0}{m\omega^2}\left(\cos(\omega t + \alpha) - \cos(\omega t0 + \alpha)\right) \tag{3.1}$$

In this case, the mass m and charge e refer to the rest mass and electric charge of the electron, respectively, and $E_0\sin(\omega t + \alpha)$ represents the RF electric field applied between the plates. The initial position and velocity of the electron at time t_0 are denoted by y_0 and v_0 respectively. For this particular scenario, we set $t_0 = 0$, $v_0 = \frac{1}{5}$, $y_0 = 0\omega = 1$, $\alpha = 0$ and $e\frac{E_0}{m} = 1$. This results in the transformation of equation (3.1) into a nonlinear equation of the form

$$F_3(r) = \frac{r}{5} + \cos(r) - \frac{\pi}{2}.$$

This function exhibits a zero, which is $\alpha \approx 5.258243$ The initial approximation of the function's root, has been selected: $r_0 = 5$.
Based on the results obtained, shown in Table 3, it is clear that the proposed methods are, in general, faster than the existing ones.

TABLE III: NUMERICAL OUTCOMES DERIVED FROM EXAMPLE 3.3 USING DIFFERENT METHODOLOGIES FOR INSTANCE.

Method	$\lvert r_{(3)} - r_{(2)}\rvert$	$\lvert r_{(4)} - r_{(3)}\rvert$	$\lvert r_{(5)} - r_{(4)}\rvert$	$F(r_{(5)})$	ACOC	CPU Timing
PM_1	5.9(−18)	1.1(−70)	1.8(−281)	1.1(−1124)	4.0000	0.468
PM_2	1.3(−17)	3.4(−69)	1.2(−275)	2.5(−1101)	4.0000	0.281
PM_3	1.9(−17)	1.4(−68)	4.2(−273)	3.6(−1091)	4.0000	0.391
AC_1	6.1(−15)	3.8(−58)	5.5(−231)	2.7(−922)	4.0000	1.312
AC_2	7.6(−15)	1.0(−57)	3.5(−229)	4.6(−915)	4.0000	1.156
FC_1	1.1(−16)	3.7(−70)	9.2(−280)	3.7(−1118)	4.0000	1.675
FC_1	9.3(−18)	1.7(−65)	1.0(−260)	1.2(−1041)	4.0000	1.875

IV. CONCLUDING SUMMARY

This study presents a novel and efficient family of fourth-order derivative-free iterative methods for solving nonlinear equations. The proposed methods utilize the weight function approach, enabling the development of new or existing iterative methods through the selection of appropriate weight functions. Computational results demonstrate that the proposed methods exhibit a higher convergence rate compared to other methods of the same order at the tested points.
Additionally, our methods deliver more accurate results in situations where other methods face challenges in terms of convergence rate and error reduction.

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