

A Note on Cartesian Product $k_2 \times k_g$ Graphs

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Abstract—In this article we have computed the greater bounds for the Kirchhoff eigenvalues of the Cross product of graph $k_2 \times k_g$ has been computed. Furthermore, we have determined the greater bound for the addition of the all eigenvalues of $k_2 \times k_g$ and Energy of Graph

Keywords— Cross product of Graph, Kirchhoff matrix, largest eigenvalue, Trace of a matrix, Greater bound.

Mathematics Topic Classification: 05C50, 15A18, 05C22. **Keywords:** spectral radius; Adjacency Matrix; Kirchhoff Matrix.

I. INTRODUCTION

For all phrases and notes of graph theory, bookworms must refer to [2]. Here we consider simple and non-loop graphs including the restricted number of nodes and edges.

Let $G = (V, E)$ be a connected graph with t nodes and s edges. Then $L = D - A$ is the Kirchhoff matrix of G , where D and A are diagonal degrees.

Matrix and the adjacency matrix of G respectively. It is recognized that L is positive semi definite, Symmetric and $|L|=0$.

Let $\alpha_1(G) \geq \alpha_2(G) \geq \dots \geq \alpha_{n-1}(G) \geq \alpha_n(G) = 0$ are the eigenvalues Of L . The Kirchhoff eigenvalues of a graph are significant in the graph theory since They have a near relation to frequent graph invariants.

Several new inequalities are obtained for the modules, the real part, and imaginary part of a linear arrangement of the ordered eigenvalues of a square matrix.

The spectrum of a graph G involves of its eigenvalues, with the leading eigenvalue α_1 mentioned to as the spectral limit or index of G . In spectral graph theory, spectral inequalities involve one or more spectral invariants, typically eigenvalues, and play a crucial role in describing a graph's structure. These inequalities have diverse applications and are derived using various methods, including the Rayleigh principle, eigenvalue interlacing, spectral moments, eigenvector analysis, and relationships between spectral and structural invariants. A specific type of spectral inequality is a bound on a spectral invariant, which may apply to entire graphs or a particular class of graphs. In equally belongings, extremal graphs for a given invariant are those that achieve the bound, if it exists. For instance, it is established that $\alpha_1 \geq 0$, with equality occurring only when G is absolutely disconnected. (Lower or greater)

In this article we gave a different greater bound for the Kirchhoff Eigen values of a Cross product of Graph. Let G be a unassuming graph on t nodes and s edges where $h=2+g$, and the values of g lies in, $2 \leq g \leq n$ in complete graph K_g .

Let G be a simple graph with vertex set $v(G)$ and the edge set $E(G)$. and let $s_k(G) = \sum_{i=1}^n \alpha_i(G)$. Let d_v be the degree of v in G .

Let $d_n(G) = |\{v_1 \in V(G) : d_{v_1} \geq k\}|$ for $k=1, 2, 3, \dots, n$. visibly

$$d_1(G) \geq d_2(G) \geq d_3(G) \dots \geq d_n(G).$$

In [6] the Grone –Merries conjectures states that $s_k(G) \leq \sum_{i=1}^k d_i^*(G)$ it was proven by Bai [1]. The following are the new upper bounds of graphs in different ways.

The energy $E(G)$ of a graph G is well-defined to be the addition of the unconditional values of its Eigen values. Hence if $A(G)$ is the adjacency matrix of G . and $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ are Eigen values of $A(G)$, then E

$$(G) = \sum_{i=1}^n |\alpha_i|$$

Theorem 1: if $k_2 \times k_g$ be the Cartesian product of the Graph. Then the energy of $k_2 \times k_g$ is

$$\text{Energy } [k_2 \times k_g] = 4(g-1)$$

Proof: The Adjacency matrix is

$$A(k_2 \times k_g) = \begin{bmatrix} A(k_g) & I_g \\ I_g & A(k_g) \end{bmatrix}$$

The characteristic polynomial is

$$ch_{k_2 \times k_g}(x) = (x+2)^{g-1} + (x-g)(x-(g-2))x^{g-1}$$

$$\text{Spec } [k_2 \times k_g] = \begin{bmatrix} -2 & g & g-2 & 0 \\ g-1 & 1 & 1 & g-1 \end{bmatrix}$$

$$\text{Energy } [k_2 \times k_g] = |-2|(g-1) + |g|(1) + |(g-2)|(1) + |0|(g-1)$$

$$= 2(g-1) + g + g - 2$$

$$= 2g - 2 + g + g - 2$$

$$= 4g - 4$$

$$\text{Energy } [k_2 \times k_g] = 4(g-1)$$

Theorem 2: Let the $k_2 \times k_g$ be the Cross product of the Graph with t nodes and s edges then the trace of the Laplacian matrix is $s_k(k_2 \times k_g) = 2g^2$

Proof: we have

$$s_k(k_2 \times k_g) = (g-1)h + (g-1)(h-2) + 2 + 0$$

$$= (g-1)[h + (h-2)] + 2$$

$$= (g-1)[2h-2] + 2$$

$$= 2(g-1)(h-1) + 2$$

$$= 2[(g-1)(h-1) + 1]$$

$$= 2[(g-1)(g+2-1) + 1]$$

$$= 2[(g-1)(g+1) + 1]$$

$$= 2[g^2 - 1 + 1] = 2g^2$$

$$s_k(k_2 \times k_g) = 2g^2$$

Propositions 3: Let the $k_2 \times k_g$ be the Cross product of the Graph with t nodes and s edges then the greater bounds is

$$\alpha_1 \leq \frac{2}{g} \left[\frac{(g-1)^{\frac{\sqrt{5}}{4}} [h^{\frac{\sqrt{5}}{4}} + (h-2)^{\frac{\sqrt{5}}{4}}] + 2^{\frac{\sqrt{5}}{4}}}{4} \right] + 2.$$

Proof: We have,

$$\begin{aligned}
& \left[\frac{(g-1)\sqrt{\frac{5}{4}}h\sqrt{\frac{5}{4}} + (g-1)\sqrt{\frac{5}{4}}(h-2)\sqrt{\frac{5}{4}} + 2\sqrt{\frac{5}{4}} + 0}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \geq \left[\frac{(g-1)h + (g-1)(h-2) + 2 + 0}{4} \right] \\
& \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}} + 0}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \geq \frac{s_k(k_2 \times k_g)}{4} \\
\text{By Theorem 2, } \frac{2g^2}{4} & \leq \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \\
g & \leq \frac{2}{g} \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \\
g + 2 & \leq \frac{2}{g} \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} + 2 \\
\alpha_1 & \leq \frac{2}{g} \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} + 2
\end{aligned}$$

Propositions 4: Let the $k_2 \times k_g$ be the Cross product of the Graph with t nodes and s edges ,then the greater bounds for the addition of the Eigen values of the graph $k_2 \times k_g$ is

$$s_k(k_2 \times k_g) \leq 4 \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}}$$

Proof: we have

$$\begin{aligned}
& \left[\frac{(g-1)\sqrt{\frac{5}{4}}h\sqrt{\frac{5}{4}} + (g-1)\sqrt{\frac{5}{4}}(h-2)\sqrt{\frac{5}{4}} + 2\sqrt{\frac{5}{4}} + 0}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \geq \left[\frac{(g-1)h + (g-1)(h-2) + 2 + 0}{4} \right] \\
& \frac{s_k(k_2 \times k_g)}{4} \leq \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}} \\
& s_k(k_2 \times k_g) \leq 4 \left[\frac{(g-1)\sqrt{\frac{5}{4}}[h\sqrt{\frac{5}{4}} + (h-2)\sqrt{\frac{5}{4}}] + 2\sqrt{\frac{5}{4}}}{4} \right]^{\frac{1}{\sqrt{\frac{5}{4}}}}
\end{aligned}$$

II. CONCLUSION

In this article, we used the trace of the Kirchhoff matrix of graph to determine the greater bounds of the eigenvalues and the greater bounds for the addition of the Eigen values (trace of a matrix) of the Kirchhoff matrix of Cross product of graph. In addition to calculate the Energy of graph.

Values of Greater bounds for the Eigen value α_1 of Kirchhoff matrix of Cross Product of Graph $k_2 \times k_g$ as follows:

TABLE I

Graph $k_2 \times k_g$	Values of h	Eigen value α_1	Values of upper bounds
$k_2 \times k_2$	h=4	$\alpha_1 = 4$	4.076
$k_2 \times k_3$	h=5	$\alpha_1 = 5$	5.1505
$k_2 \times k_4$	h=6	$\alpha_1 = 6$	6.2333
$k_2 \times k_5$	h=7	$\alpha_1 = 7$	7.3161
$k_2 \times k_6$	h=8	$\alpha_1 = 8$	8.3979
$k_2 \times k_7$	h=9	$\alpha_1 = 9$	9.4788
$k_2 \times k_8$	h=10	$\alpha_1 = 10$	10.5589

Values of Greater bounds for the sum of the Eigen value (Trace of a matrix) of Kirchhoff matrix of Cross Product of Graph $k_2 \times k_g$ as follows:

TABLE II

Graph $k_3 \times k_g$	Values of h	$s_k(k_3 \times k_g)$	Values of upper bounds
$k_2 \times k_2$	h=4	8	8.3042
$k_2 \times k_3$	h=5	18	18.9032
$k_2 \times k_4$	h=6	32	33.4666
$k_2 \times k_5$	h=7	50	53.161
$k_2 \times k_6$	h=8	72	76.775

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