Grenze International Journal of Engineering and Technology, Jan Issue

A Dynamic Formulation of Sophisticated High-Speed Rail Wagon with Rail-Structure Interaction for the Bridge Structure

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***Abstract*—India is on the root to promote the elevated root of transportation. Bridges, flyovers are serving the same moto and they support the widest load variety, earlier bridges were used only for pedestrian and vehicular loads but as time passes it uses more intended towards mass transportation with high speed. When a bridge is designed to serve such heavy and rapid load; it should be analyzed for such load intensity. The question is how this high-speed rail load should be applied? The trainload is applied to bridge structure by different approaches by different researchers. In this article, the 10-degree freedom system rail wagon is prepared with the primary and secondary suspension system. Interactive equations are developed for rail and the bridge structure interaction. The series of rail axle loads as a result of 10 DOF rail wagon gives a satisfactory result.**

***Index Terms*— Rail Bridge, Dynamic High-Speed Rail wagon, rail structure interaction.**

I. INTRODUCTION

As time evolved infrastructure facilities are its top and it's a matter of global prestige for any nation. With the development of technologies and economies, the need for a high-speed transport network arises. The high-speed rail network is the most promising way of mass transportation which will reduce traveling time to a certain extent. Over the world, the demand is raised for the development of rail supporting structures. High-speed railway is accepted broadly in many developed nations but by introducing the Mumbai-Ahmedabad high-speed rail corridor, India also enters the same field. As a result of such evolution more complex structure comes in reality. The vehicle used for bridge analysis plays a vital role in bridge behaviour forecasting. The more detailed the rail model, the more accurate the results will be in terms of passenger comfort and serviceability criteria. With the development of high-speed rail, different researchers used different rail load application approaches. Earlier The moving axle load model has been used by the researchers although the moving axle load models are suitable for only those bridges where only bridge dynamic response is required. [1-3] then the moving mass approach comes in reality when vehicle mass ratio cannot be neglected and the vehicle inertial effects are very accentual. [4] Afterward, the spring-mass model for the Rail vehicle has been developed by many authors Which consists of the suspension system of the vehicle epitomized by springs and dashpots. This modelling approach is the primitive modelling approach with a single degree freedom system. [5, 6, 9] Recently, There is the most realistic and detailed modelling approach used to simulate the vehicle. This model comprised of carriage, bogies, and wheelset modeled as 2 masses and connected by suspension systems (primary and secondary spring and

Grenze ID: 01.GIJET.7.1.525

dampers). The difficulty level of the vehicle model used by researchers varies from four Degree of freedom (D.O.F.) to twenty-seventh degree of freedom (D.O.F.) according to passenger safety, comfort criteria, and analysis requirements [7-19]. The complexity of such a model is higher than the above mention approaches. In the present study, a 10-degree freedom system rail wagon is modelled with rail- structure interaction.

II. DYNAMIC FORMULATION

A. Modelling of the train model

A typical railway vehicle consists of the vehicle body carrying passengers or freight and the wheels. The body is supported either directly on their axles or bogies. The increasingly heavy loading has called for more axles grouped into bogies, where two or more axles are mounted on the same frame. The bogie frames are connected to the axles and the railway vehicle body, respectively, through the primary and secondary suspension systems, which comprise springs and shock absorbers.

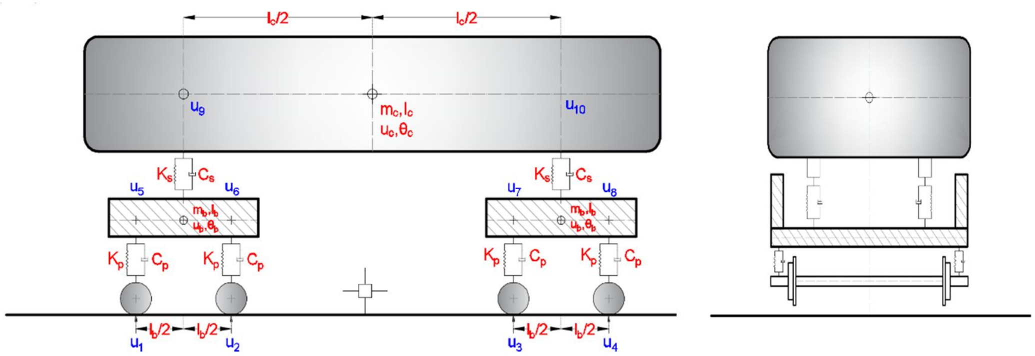

Figure 1. Model of a typical train vehicle for the analysis of vehicle-bridge interaction

The train model adopted in this study is presented in figure 1. The train carriage is modelled as a rigid body having a mass M_c and a moment of inertia I_c about the transverse horizontal axis through its centroid. Similarly, each bogie frame is considered as a rigid body with a mass M_b and a moment of inertia I_b about the transverse horizontal axis through its centroid. Each axle along with the wheels has a mass Mw. The spring and shock absorber in the primary suspension for each axle are characterized by spring stiffness K_P and damping coefficient C_P respectively. Similarly, the secondary suspension is defined by the spring stiffness K_s and damping coefficient C_s. The vertical displacements of the train vehicle model are described u_1 to u_{10}. The vertical displacements of the train vehicle at locations are marked by crosses in Figure 1 concerning their equilibrium positions before coming to the bridge. As the vehicle body is assumed to be rigid, its motion may be described by the vertical displacement u_c and rotation θ_c at its centroid instead of u_9 and u_{10}. Similarly, the vertical motions of the bogie frames may also be described by $u_{b1}, \theta_{b1}, u_{b2}$ and θ_{b2} at their centroid instead of u_5 to u_8. The displacement vector δ_T for the train vehicle can be written in terms of displacements at the centroid of the vehicle body $\ddot{\delta}_i^t = N_b \ddot{\delta}_b + 2v N_b^{'} \dot{\delta}_b + v^2 N_b^{''} \delta_b + a N_b^{'} \delta_b + a r^{'} + v^2 r^{''}$ and θ_C, the displacements at the centroid of the vehicle body y, the displacements at the centroids of the bogie frames $u_{b1}, \theta_{b1}, u_{b2}$, and θ_{b2} the displacements of the four axles u_1 to u_4 as $x_t = [u_C \quad \theta_C \quad u_{b1} \quad \theta_{b1} \quad u_{b2} \quad \theta_{b2} \quad u_1 \quad u_2 \quad u_3 \quad u_4]^T$ (1)

The free body diagram of carriage, bogie, and wheelset is presented in fig. 2. The internal and inertia forces are shown. The reactions at the rail and the primary and secondary suspension systems are expressed in terms of the static reaction components R_w R_p and R_s, respectively. The dynamic reaction components f, respectively, as well as the dynamic reaction components f_{c1} to f_{c4} and f_1 to f_6. The static components R_w R_p and R_s are given as,

$$R_w = M_C g / 4 + M_b g / 2 + M_w g \tag{2}$$

$$R_p = M_C g / 4 + M_b g / 2 \tag{3}$$

$$R_s = M_c g / 4 \tag{4}$$

Where g is the acceleration due to gravity. The equation of motion of each free-body can then be formulated and they are given below. As the static case is one particular case of the dynamic situation, the static reaction

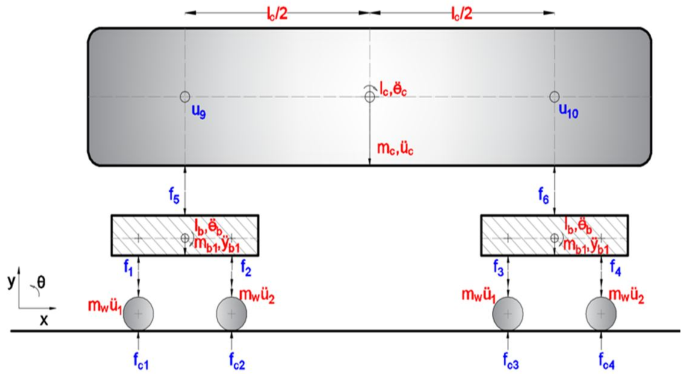

Figure 2 Internal forces acting in the 4-axle model of typical train vehicle

components cancel themselves and will not appear in these equations. The equation of motion can be represented by the mass matrix m_t, damping matrix c_t, and stiffness matrix k_t. The load vector is termed as f_t. So, the dynamic equation become,

$$m_t \ddot{x}_t + c_t \dot{x}_t + k_t x_t = f_t \tag{5}$$

Where the dot indicates differentiation concerning time t. the displacement vector x_t for the train vehicle can be divided into the upper group x_u^t, which are unconstrained by the bridge, and lower group x_i^t, which are in contact with the bridge, where

$$x_u^t = \begin{bmatrix} y_v & \theta_v & y_{f1} & \theta_{f1} & y_{f2} & \theta_{f2} \end{bmatrix}^T \tag{6}$$

$$x_i^t = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \tag{7}$$

The equation of motion for the train model can be written in terms of the sub-matrices and sub-vector as

$$\begin{bmatrix} m_{uu}^t & \\ & m_{ii}^t \end{bmatrix} \begin{Bmatrix} \ddot{x}_u^t \\ \ddot{x}_i^t \end{Bmatrix} + \begin{bmatrix} c_{uu}^t & c_{ui}^t \\ c_{iu}^t & c_{ii}^t \end{bmatrix} \begin{Bmatrix} \dot{x}_u^t \\ \dot{x}_i^t \end{Bmatrix} + \begin{bmatrix} k_{uu}^t & k_{ui}^t \\ k_{iu}^t & k_{ii}^t \end{bmatrix} \begin{Bmatrix} x_u^t \\ x_i^t \end{Bmatrix} + \begin{Bmatrix} f_u^t \\ f_i^t \end{Bmatrix} \tag{8}$$

Where the sub-matrices and sub-vectors are given below notice that the load sub-vector f_u^t is zero. The load sub-vector f_i^t comprises the transient components only as the displacements in the sub-vector x_u^t of the train are given concerning their equilibrium positions before coming to the bridge. So, the total reaction $\overline{f_i^t}$ between the train axle and the bridge has to include the static reaction.

B. Formulation of train-bridge system

To archive the above mention objective there is a need for train-bridge interaction formulation; those are followed by the following equations.

$$y(x,\ t) = y_b(x,t) + r(x) \tag{13}$$

Where, $y(x,\ t)$, $y_b(x,t)$ and $r(x)$ stands for the vertical displacement of an axle, vertical displacement of the bridge, and the rail irregularity at the same location. Which x denotes the horizontal position of the axles. The vertical displacement of the bridge at the contact point can be expressed by the nodal displacement vector. So, the final train axle load can be expressed as,

$$y(x,\ t) = N_b^e(x)\delta_b^e(t) + r(x) \tag{14}$$

The train axle may be affected by many elements so, equation number 15 can be rewritten as,

$$y(x,\ t) = N_b(x)\delta_b^e(t) + r(x) \tag{15}$$

$N_b(x)$ is the shape function matrix evaluated at the contact point between the axles and the rail. $\delta_b(t)$ is the displacement vector of the bridge and $r(x)$ is a vector containing the rail irregularities at contact points. The vertical velocity vector $\dot{\delta}_i^t$ and vertical acceleration $\ddot{\delta}_i^t$ can be expressed as,

$$\ddot{\delta}_i^t = N_b\dot{\delta}_b + vN_b^{'}\delta_b + vr' \tag{16}$$

$$\ddot{\delta}_i^t = N_b\ddot{\delta}_b + 2vN_b^{'}\dot{\delta}_b + v^2N_b^{''}\delta_b + aN_b^{'}\delta_b + ar^{'} + v^2r^{''} \tag{17}$$

Here it is noted that prime denotes the differentiation concerning x. For the ease of calculation independent variables are omitted. Putting the equations (16 – 18) into equation (8) and combining with equation (12), the equation of motion for the entire train – bridge system can be written as

$$m\ddot{x} + c\dot{x} + kx = p \tag{18}$$

Where m, c, k, p, and x represent the matrix of mass, the matrix of damping, the matrix of stiffness, load vector, and displacement vector. Equation (20) can be rewritten as

$$\begin{bmatrix} m_{uu}^t & \\ & m_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_u^t \\ \ddot{\delta}_b \end{Bmatrix} + \begin{bmatrix} c_{uu}^t & c_{tb} \\ c_{bt} & c_{bb} \end{bmatrix} \begin{Bmatrix} \dot{\delta}_u^t \\ \dot{\delta}_b \end{Bmatrix} + \begin{bmatrix} k_{uu}^t & k_{tb} \\ k_{bt} & k_{bb} \end{bmatrix} \begin{Bmatrix} \delta_u^t \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} p_u^t \\ p_b \end{Bmatrix} \tag{19}$$

In which the sub-matrices and sub-vectors are given as

$$m_{bb} = m_b + N_b^T m_{ii}^t N_b, \quad c_{tb} = c_{ui}^t N_b \tag{20, 21}$$

$$c_{bt} = N_b^T c_{iu}^t, \quad c_{bb} = c_b + 2\nu N_b^T m_{ii}^t N_b^t + N_b^T c_{ii}^t N_b \tag{22, 23}$$

$$k_{tb} = \nu c_{ui}^t N_b^{'} + k_{ui}^t N_b, \quad k_{bt} = N_b^T k_{iu}^t \tag{24, 25}$$

$$k_{bb} = k_b + \nu^2 N_b^T m_{ii}^t N_b^{''} + a N_b^T m_{ii}^t N_b^{'} + \nu N_b^T c_{ii}^t N_b^{'} + N_b^T k_{ii}^t N_b \tag{26}$$

$$p_u^t = -(\nu c_{u1}^t r^{'} + k_{u1}^t r), \quad p_b = -N_b^T (R_w + a m_{ii}^t r^{'} + \nu^2 m_{ii}^t r^{''} + \nu c_{ii}^t r^{'} + k_{ii}^t r) \tag{27, 28}$$

The problem is solved in the Matlab Simulink programme by the newton Rapson method. However, the matrices and vectors in the equation of motion for the train- bridge interaction is time dependant. The direction and position of the train axle are the key parameters in the analysis.

III. Validation of Train Model

With the purpose of accuracy verification of the 10 DOF train model, the simply-supported bridge is analyzed under an ETR500Y train, shown in Fig. 4, the analysis result is compared with the result in Liu et al. [9] The basic geometric data of the bridge are, span (L) = 34 m, with a line density $\bar{m}$ = 11400 kg/m and bending stiffness EI = 9.92×1010 N·m^2. The bridge has a modal damping ratio of 2%. A simplified ETR500Y train model is applied which is composed of a locomotive followed by eight passenger cars and another locomotive. The properties of the ETR500Y train model are shown in table I.

TABLE I. CHARACTERISTICS OF ETR500Y HIGH-SPEED TRAIN [10]

Item	Unit	Locomotive	Passenger Car
Length of carriage (l_c)	m	19.7	26.1
Length of bogie (l_b)	m	3.6	3.6
Mass of carriage (m_c)	KN	548.94	335.69
Mass of bogie (m_b)	KN	38.21	27.07
Mass of wheel (m_w)	kg	20.19	15.52
Mass moment of inertia of carriage @ z-axis (I_{zc})	KN.m^2	1630.52	1760.62
Mass moment of inertia of bogie @ z-axis (I_{zc})	KN.m^2	8.107	4.071
Mass moment of wheel set (I_w)	KN.m^2	1.164	0.753
Vertical stiffness [primary suspension] (K_p)	KN/m	896100	404370
Vertical damping [primary suspension] (C_p)	KN.s/m	7625	3750
Vertical stiffness [secondary suspension] (K_s)	KN/m	236030	90277
Vertical damping [secondary suspension] (Cs)	KN.s/m	18125	8125
Distance between two centers of bogie (l_b)	m	12	19
Distance between two centers of axles (l_a)	m	3	3
The overall length of carriage (m)	m	19.7	26.1

As per the equation outcome, the average static axle load for the locomotives and passenger cars is 176.4 and 113.01 KN, respectively.

The impact factor of vertical displacement at the bridge center at various train speed ratios is presented in Figure 5. The impact factor is commonly used to show the effect of trainload on the bridge which is defined as, Impact

$$\text{factor}=\frac{R_{dyn}^{max} - R_{sta}^{max}}{R_{sta}^{max}} \tag{29}$$

Where, R_{static} and $R_{dynamic}$ are the maximum dynamic and static responses of the bridge under the trainload. Note that the responses can be either the displacement at bridge element nodes or the tension of a cable. The maximum static response of the bridge is obtained by placing the self-weight load of the train in various positions on the deck and calculating the maximum value of the responses. The speed ratio is defined as,

$$\alpha = {}^{v}/_{f_1 l_t} \tag{30}$$

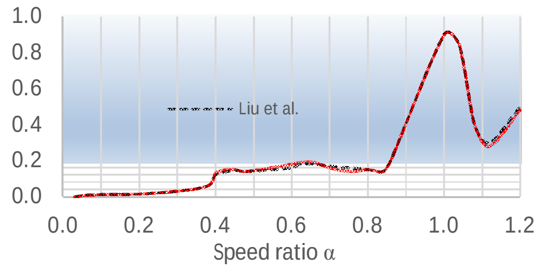

Figure 3 The impact factor of vertical displacement at the bridge center for the various train speed ratios

The fundamental frequency of the bridge is f1=4.01 Hz and v is the train speed. Lt is the length of a passenger car. The analysis results in terms of impact factor obtained in this study match well with that in Liu et al. (2009), so the train model and dynamic analysis procedures which are used in this study are reliable.

IV. CONCLUSION

The dynamic modelling for high-speed rail wagon with a 10-degree freedom system provides all the basic information about passenger serviceability criteria.

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APPENDIX A

Sign convention adopted for the equation formulation are, for the displacement (Positive in gravity direction) Rotation (Positive in clockwise direction). The equation of motion of carriage can be written as:

$m_c \ddot{u}_c + f_5 + f_6 = 0$ (1)

$I_C \ddot{\theta}_C + (f_6 - f_5) l_c / 2 = 0$ (2)

Similarly, the equation of motion of bogie 1 can be written as:

$m_{b1} \ddot{u}_{b1} + f_1 + f_2 - f_5 = 0$ (3)

$I_{b1} \ddot{\theta}_{b1} + (f_2 - f_1) l_{b1} / 2 = 0$ (4)

The equation of motion of bogie 2 can be written as:

$m_{b2} \ddot{u}_{b2} + f_1 + f_2 - f_5 = 0$ (5)

$I_{b2} \ddot{\theta}_{b2} + (f_2 - f_1) l_{b2} / 2 = 0$ (6)

The equation of motion of wheel axle can be written as:

$m_w \ddot{u}_i - f_i + f_{ci} = 0$ (For i = 1 to 4) (7)

Displacement at the center of bogie frame, $u_{b1}, \theta_{b1}, u_{b2}$ *and* θ_{b2} as

$u_5 = u_{b1} - \theta_{b1} l_b / 2, \; u_8 = u_{b2} - \theta_{b2} l_b / 2,$ (8 to 13)
$u_6 = u_{b1} + \theta_{b1} l_b / 2, \; u_9 = u_c - \theta_c l_c / 2,$
$u_7 = u_{b2} - \theta_{b2} l_b / 2, \; u_{10} = u_c + \theta_c l_c / 2,$

The dynamic reaction component f1 to f6 can there for be written as,

$f_1 = k_p (u_{b1} - \theta_{b1} l_b / 2 - u_1) + c_p (\dot{u}_{b1} - \dot{\theta}_{b1} l_b / 2 - \dot{u}_1),$
$f_2 = k_p (u_{b1} + \theta_{b1} l_b / 2 - u_2) + c_p (\dot{u}_{b1} + \dot{\theta}_{b1} l_b / 2 - \dot{u}_2),$
$f_3 = k_p (u_{b2} - \theta_{b2} l_b / 2 - u_3) + c_p (\dot{u}_{b2} - \dot{\theta}_{b2} l_b / 2 - \dot{u}_3),$ (14 to 19)
$f_4 = k_p (u_{b2} + \theta_{b2} l_b / 2 - u_4) + c_p (\dot{u}_{b2} + \dot{\theta}_{b2} l_b / 2 - \dot{u}_4),$
$f_5 = k_s (u_c - \theta_c l_c / 2 - u_{b1}) + c_s (\dot{u}_{c1} - \dot{\theta}_{c1} l_c / 2 - \dot{u}_{b1}),$
$f_6 = k_s (u_c + \theta_c l_c / 2 - u_{b2}) + c_s (\dot{u}_{c1} + \dot{\theta}_{c1} l_c / 2 - \dot{u}_{b2}),$

Substituting eq. 14 to 19 into eq. 1 to 7. The equation of motion of train becomes,

$m_c \ddot{u}_c + 2 c_s \dot{u}_c - c_s (\dot{u}_{b1} + \dot{u}_{b2}) + 2 k_s u_c - k_s (u_{b1} + u_{b2}) = 0$ (20)

$I_c \ddot{\theta}_C + (c_s {l_c}^2 / 2) \dot{\theta}_c + (c_s l_s / 2)(\dot{u}_{b1} - \dot{u}_{b2}) + (k_s {l_c}^2 / 2) \theta_c + (k_s l_s / 2)(u_{b1} - u_{b2}) = 0$ (21)

$m_{b1} \ddot{u}_{b1} - c_s \dot{u}_c + (c_s + 2 c_p) \dot{u}_{b1} + (c_s l_s / 2) - c_p (\dot{u}_1 + \dot{u}_2) - k_s u_c + (k_s l_c / 2) \theta_c$
$+ (k_s + 2 k_p) u_{b1} - k_p (u_1 + u_2) = 0$ (22)

$I_{b1} \ddot{\theta}_{b1} + (c_p {l_b}^2 / 2) \dot{\theta}_{b1} + (c_p l_b / 2)(\dot{u}_1 - \dot{u}_2) + (k_p {l_b}^2 / 2) \theta_{b1} + (k_p l_b / 2)(u_1 - u_2) = 0$ (23)

$m_{b2} \ddot{u}_{b2} - c_s \dot{u}_c + (c_s + 2 c_p) \dot{u}_{b2} - (c_s l_s / 2) - c_p (\dot{u}_3 + \dot{u}_4) - k_s u_c - (k_s l_c / 2) \theta_c$
$+ (k_s + 2 k_p) u_{b2} - k_p (u_3 + u_4) = 0$ (24)

$I_{b2} \ddot{\theta}_{b2} + (c_p {l_b}^2 / 2) \dot{\theta}_{b2} + (c_p l_b / 2)(\dot{u}_3 - \dot{u}_4) + (k_p {l_b}^2 / 2) \theta_{b2} + (k_p l_b / 2)(u_3 - u_4) = 0$ (25)

$m_w \ddot{u}_1 - c_p \dot{u}_{b1} + (c_p l_b / 2) \dot{\theta}_{b1} + c_p \dot{u}_1 - k_p u_{b1} + (k_p l_b / 2) \theta_{b1} + k_p u_1 - f_{c1} = 0$ (26)

$m_w \ddot{u}_2 - c_p \dot{u}_{b1} - (c_p l_b / 2) \dot{\theta}_{b1} + c_p \dot{u}_2 - k_p u_{b1} - (k_p l_b / 2) \theta_{b1} + k_p u_2 - f_{c2} = 0$ (27)

$m_w \ddot{u}_3 - c_p \dot{u}_{b2} + (c_p l_b / 2) \dot{\theta}_{b2} + c_p \dot{u}_3 - k_p u_{b2} + (k_p l_b / 2) \theta_{b2} + k_p u_3 - f_{c3} = 0$ (28)

$m_w \ddot{u}_4 - c_p \dot{u}_{b2} - (c_p l_b / 2) \dot{\theta}_{b2} + c_p \dot{u}_4 - k_p u_{b2} - (k_p l_b / 2) \theta_{b2} + k_p u_4 - f_{c4} = 0$ (29)

Equation 20 to 29 can be written in matrix notation in terms of mass matrix m, damping matrix c, the stiffness matrix k and the load vector f for the train as

$m \ddot{x} + c \dot{x} + k x = f$ (30)

Where the dot denotes differentiation with respect to time t. the displacement vector at for the train vehicle can be divided into upper group ${a^t}_u$, which are unconstraint by the bridge, and the lower group ${a^t}_i$, which are in contact with the bridge, where,

$$a^t{}_u = [u_c \quad \theta_c \quad u_{b1} \quad \theta_{b1} \quad u_{b2} \quad \theta_{b2}]^T \tag{31}$$
$$a^t{}_i = [u_1 \quad u_2 \quad u_3 \quad u_4]^T \tag{32}$$
The equation of motion of train model can be written in terms of the submatrices and sub-vectors as,
$$\begin{bmatrix} m^t{}_{uu} & \\ & m^t{}_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{\delta}^t{}_u \\ \ddot{\delta}^t{}_i \end{Bmatrix} + \begin{bmatrix} c^t{}_{uu} & c^t{}_{ui} \\ c^t{}_{iu} & c^t{}_{ii} \end{bmatrix} \begin{Bmatrix} \dot{\delta}^t{}_u \\ \dot{\delta}^t{}_i \end{Bmatrix} + \begin{bmatrix} k^t{}_{uu} & k^t{}_{ui} \\ k^t{}_{iu} & k^t{}_{ii} \end{bmatrix} \begin{Bmatrix} \delta^t{}_u \\ \delta^t{}_i \end{Bmatrix} + \begin{Bmatrix} f^t{}_u \\ f^t{}_i \end{Bmatrix} \tag{33}$$
Where, the submatrices and sub-vectors are given below,
$$m^t{}_{uu} = diag[m_c \quad I_c \quad m_{b1} \quad I_{b1} \quad m_{b2} \quad I_{b2}] \tag{34}$$
$$m^t{}_{ii} = diag[m_w \quad m_w \quad m_w \quad m_w] \tag{35}$$
$$c^t{}_{uu} = \begin{bmatrix} 2c_s & 0 & -c_s & 0 & -c_s & 0 \\ 0 & c_s l_c{}^2/2 & c_s l_c/2 & 0 & -c_s l_c/2 & 0 \\ -c_s & c_s l_c/2 & c_s + 2c_p & 0 & 0 & 0 \\ 0 & 0 & 0 & c_p l_b{}^2/2 & 0 & 0 \\ -c_s & -c_s l_c/2 & 0 & 0 & c_s + 2c_p & 0 \\ 0 & 0 & 0 & 0 & 0 & c_p l_b{}^2/2 \end{bmatrix} \tag{36}$$
$$c^t{}_{ui} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -c_p & -c_p & 0 & 0 \\ c_p l_b/2 & -c_p l_b/2 & 0 & 0 \\ 0 & 0 & -c_p & -c_p \\ 0 & 0 & c_p l_b/2 & -c_p l_b/2 \end{bmatrix} \tag{37}$$
$$c^t{}_{ii} = diag[c_p \quad c_p \quad c_p \quad c_p] \tag{38}$$
$$c^t{}_{uu} = \begin{bmatrix} 2k_s & 0 & -k_s & 0 & -k_s & 0 \\ 0 & k_s l_c{}^2/2 & k_s l_c/2 & 0 & -k_s l_c/2 & 0 \\ -k_s & k_s l_c/2 & k_s + 2k_p & 0 & 0 & 0 \\ 0 & 0 & 0 & k_p l_b{}^2/2 & 0 & 0 \\ -k_s & -k_s l_c/2 & 0 & 0 & k_s + 2k_p & 0 \\ 0 & 0 & 0 & 0 & 0 & k_p l_b{}^2/2 \end{bmatrix} \tag{39}$$
$$c^t_{ii} = diag[c_2 \quad c_2 \quad c_2 \quad c_2] \tag{40}$$
$$k^t_{uu} = \begin{bmatrix} 2k_1 & 0 & -k_1 & 0 & -k_1 & 0 \\ 0 & k_1 l_1^2/2 & k_1 l_1/2 & 0 & -k_1 l_1/2 & 0 \\ -k_1 & k_1 l_1/2 & k_1 + 2k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 l_2^2/2 & 0 & 0 \\ -k_1 & -k_1 l_1/2 & 0 & 0 & k_1 + 2k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_2 l_2^2/2 \end{bmatrix} \tag{41}$$
$$k^t_{ui} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2 & -k_2 & 0 & 0 \\ k_2 l_2/2 & -k_2 l_2/2 & 0 & 0 \\ 0 & 0 & -k_2 & -k_2 \\ 0 & 0 & k_2 l_2/2 & -k_2 l_2/2 \end{bmatrix} = (k^t_{iu})^T \tag{42}$$