

On Eigenvalues with Multiplicity

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Abstract—When the characteristic equation of a matrix A has distinct roots, the technique of Faddeev-Sominsky determines the characteristic vectors (eigenvectors) corresponding to the roots. In this article, we demonstrate how to construct the eigenvectors linked to an eigenvalue with multiplicity using Gower's approach.

Index Terms— Eigenvalue problem, Faddeev-Sominsky's algorithm, Gower's process, Minimal polynomial, Leverrier-Takeno's method.

I. INTRODUCTION

Let A denotes an $n \times n$ matrix. We begin by considering the basic eigenvalue problem [1]:

$$A \vec{u}_k = \lambda_k \vec{u}_k, \quad (1)$$

in which the proper values of λ_k are nothing but the zeros of $p(s)$, the characteristic polynomial of the matrix A . We write

$$p(s) = s^n + a_1 s^{n-1} + \dots + a_n, \quad p(\lambda_k) = 0, \quad (2)$$

whose coefficients a_j can be obtained via Leverrier-Takeno's process [2-6]:

$$r a_r + a_{r-1} s_1 + a_{r-2} s_2 + \dots + a_1 s_{r-1} + s_r = 0, \quad a_0 = 1, \quad s_r = \text{trace } A^r, \quad r = 1, 2, \dots, n, \quad (3)$$

that is

$$\begin{aligned} a_1 &= -s_1, & 2! a_2 &= (s_1)^2 - s_2, & 3! a_3 &= -(s_1)^3 + 3 s_1 s_2 - 2 s_3, \\ 4! a_4 &= (s_1)^4 - 6 (s_1)^2 s_2 + 8 s_1 s_3 + 3 (s_2)^2 - 6 s_4, & & \text{etc.} \end{aligned} \quad (4)$$

Note that $\det A = (-1)^n a_n$. The quantities a_j can be expressed using the well-known complete Bell polynomials [7, 8].

However, we also have the algorithm of Faddeev-Sominsky [6, 9-14]:

$$\begin{aligned} B_1 &= A, & q_1 &= \text{tr } B_1, & C_1 &= B_1 - q_1 I, \\ B_2 &= C_1 A, & q_2 &= \frac{1}{2} \text{tr } B_2, & C_2 &= B_2 - q_2 I, \end{aligned}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \mathbf{B}_{n-1} = \mathbf{C}_{n-2} \mathbf{A}, & q_{n-1} = \frac{1}{n-1} \operatorname{tr} \mathbf{B}_{n-1}, & \mathbf{C}_{n-1} = \mathbf{B}_{n-1} - q_{n-1} \mathbf{I}, \\ \mathbf{B}_n = \mathbf{C}_{n-1} \mathbf{A}, & q_n = \frac{1}{n} \operatorname{tr} \mathbf{B}_n, & \end{array} \quad (5)$$

such that $q_j = -a_j$.

The theorem of Cayley-Hamilton-Frobenius [1] demonstrates that the matrix $\mathbf{A}$ satisfies its characteristic equation (2):

$$\mathbf{A}^n + a_1 \mathbf{A}^{n-1} + \dots + a_{n-1} \mathbf{A} + a_n \mathbf{I} = \mathbf{0}. \quad (6)$$

If $a_n \neq 0$, then $\mathbf{A}$ is a non-singular matrix and from equations (5) and (6), we obtain the inverse matrix of $\mathbf{A}$ as

$$\mathbf{A}^{-1} = -\frac{1}{a_n} (\mathbf{A}^{n-1} + a_1 \mathbf{A}^{n-2} + \dots + a_{n-1} \mathbf{I}) = \frac{1}{q_n} \mathbf{C}_{n-1}. \quad (7)$$

If $\mathbf{A}$ is a singular matrix, then the corresponding adjoint matrix is expressed as (see [10])

$$\operatorname{Adj} \mathbf{A} = (-1)^{n+1} \mathbf{C}_{n-1}. \quad (8)$$

From equation (5), we have the following Faddeev's matrices [15]:

$$\mathbf{C}_k = \mathbf{A}^k + a_1 \mathbf{A}^{k-1} + a_2 \mathbf{A}^{k-2} + \dots + a_{k-1} \mathbf{A} + a_k \mathbf{I}, \quad k = 1, 2, \dots, n-1. \quad (9)$$

Therefore, equations (6) and (9) imply the relation

$$\mathbf{C}_n = \mathbf{B}_n - q_n \mathbf{I} = \mathbf{0}.$$

If the roots of equation (2) are all distinct, then the method of Faddeev-Sominsky permits us to derive the corresponding proper vectors of $\mathbf{A}$, which is possible because for a given k , every column of

$$\mathbf{Q}_k \equiv \lambda_k^{n-1} \mathbf{I} + \lambda_k^{n-2} \mathbf{C}_1 + \dots + \mathbf{C}_{n-1}, \quad k = 1, \dots, n, \quad (10)$$

verifies (1); that is

$$\mathbf{A} \mathbf{Q}_k = \lambda_k \mathbf{Q}_k. \quad (11)$$

Hence $\mathbf{Q}_k \neq \mathbf{0}$, $\forall k$ and all columns of $\mathbf{Q}_k$ are proportional to each other. Therefore $\operatorname{rank} \mathbf{Q}_k = 1$ (See [16]). In fact, from [15, 17-19], we have

$$\mathbf{Q}(s) \equiv s^{n-1} \mathbf{I} + s^{n-2} \mathbf{C}_1 + s^{n-3} \mathbf{C}_2 + \dots + \mathbf{C}_{n-1} = \frac{p(s)}{s \mathbf{I} - \mathbf{A}}, \quad (12)$$

with $\mathbf{Q}(\lambda_k) = \mathbf{Q}_k$, which enables us to obtain equation (11).

Now, we employ equation (12) to an eigenvector of the matrix $\mathbf{A}$ to get

$$\mathbf{Q}(s) \vec{u}_k = \frac{p(s)}{s - \lambda_k} \vec{u}_k. \quad (13)$$

Therefore $\mathbf{Q}_k \vec{u}_k = \vec{0}$, $j \neq k$, when all the roots of $p(s)$ are distinct, and

$$\mathbf{Q}_k \vec{u}_k = \left[\lim_{s \rightarrow \lambda_k} \frac{p(s)}{s - \lambda_k} \right] \vec{u}_k = \left[\prod_{j=1, j \neq k}^n (\lambda_k - \lambda_j) \right] \vec{u}_k. \quad (14)$$

Thus $\mathbf{Q}_k$ is not null and the vector $\vec{u}_k$ is also its eigenvector. We note the fact that the algorithm of Leverrier-Takeno-Faddeev-Sominsky was better improved by the researchers Souriau [20] and Frame [21] (See also [22, 23]).

However, if the eigenvalue λ_k has multiplicity >1 , equation (14) offers $\mathbf{Q}_k \vec{u}_k = \vec{0}$, in which case $\mathbf{Q}_k$ may be null giving no details about the related eigenvectors. In the next section, we shall present the Gower's technique [16] to compute proper eigen vectors related to an eigenvalue with multiplicity >1 .

II. GOWER'S TECHNIQUE

We begin by accepting that λ_k is an eigenvalue of the matrix $\mathbf{A}$ with multiplicity. Let the minimal polynomial (See [1, 24]) of the matrix $\mathbf{A}$ be

$$b(s) = s^m + b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m, \quad m \leq n, \quad b_0 = 1. \quad (15)$$

Then the columns of the matrix

$$\mathbf{E}_k = \begin{cases} \mathbf{M}_{m-2} + b_{m-1} \mathbf{I}, & \lambda_k = 0, \\ \lambda_k^{m-1} \mathbf{M}_0 + \lambda_k^{m-2} \mathbf{M}_1 + \dots + \mathbf{M}_{m-1}, & \lambda_k \neq 0, \end{cases} \quad (16)$$

give some or all of the eigenvectors corresponding to the eigen vector λ_k (See [16]), where

$$\mathbf{M}_0 = \mathbf{A}, \quad \mathbf{M}_j = (\mathbf{M}_{j-1} + b_j \mathbf{I}) \mathbf{A}, \quad j = 1, \dots, m. \quad (17)$$

If we use the equation (17) in the equation (16) we obtain the relations

$$\mathbf{E}_k = \begin{cases} \sum_{q=0}^{m-1} b_{m-1-q} \mathbf{A}^q, & \lambda_k = 0, \\ \sum_{q=0}^{m-1} r_{m-1-q} \mathbf{A}^q, & \lambda_k \neq 0, \end{cases} \quad (18)$$

such that:

$$r_j = \sum_{t=1}^{j+1} b_{j+1-t} \lambda_k^t, \quad j = 0, \dots, m-2, \quad r_{m-1} = -b_m. \quad (19)$$

For example, if $\lambda_k = 0$, we have

$$\mathbf{E}_k = \begin{cases} \mathbf{A} + b_1 \mathbf{I}, & \text{if } m = 2; \\ \mathbf{A}^2 + b_1 \mathbf{A} + b_2 \mathbf{I}, & \text{if } m = 3; \\ \mathbf{A}^3 + b_1 \mathbf{A}^2 + b_2 \mathbf{A} + b_3 \mathbf{I}, & \text{if } m = 4, \end{cases} \quad (20)$$

and for $\lambda_k \neq 0$, we have

$$\begin{aligned} m = 2: \quad \mathbf{E}_k &= \lambda_k \mathbf{A} - b_2 \mathbf{I}; \quad m = 3: \quad \mathbf{E}_k = \lambda_k \mathbf{A} [\mathbf{A} + (\lambda_k + b_1) \mathbf{I}] - b_3 \mathbf{I}; \\ m = 4: \quad \mathbf{E}_k &= \lambda_k \mathbf{A} [\mathbf{A}^2 + (\lambda_k + b_1) \mathbf{A} + (\lambda_k^2 + \lambda_k b_1 + b_2) \mathbf{I}] - b_4 \mathbf{I}, \dots \end{aligned} \quad (21)$$

We assert that $\mathbf{E}_k$ may not generate the subspace of all independent proper vectors related to λ_k . So, it is appropriate and acceptable to amplify the technique of Gower with differentiation method (See [22, 23, 25]). In fact, when the eigen vector λ_k has multiplicity n_k , the columns of $\mathbf{Q}^{(n_k-1)}(\lambda_k)$ provide the information about the eigenvectors $\vec{u}_r$ corresponding λ_k because

$$\mathbf{Q}^{(j)}(\lambda_k) \vec{u}_r = \vec{0}, \quad j = 0, 1, \dots, n_k - 2; \quad (22)$$

and the property (22) is immediate from (13).

III. CONCLUSION

In the Gower's algorithm, the minimal polynomial is very important which can be constructed through the technique of minimized iterations of Lanczos (See [26-29]) in association with Gelbaum's method [30] or Bialas-Bialas method (See [24, 31]).

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