Intuitionistic Fuzzification of Sylow's Theorems

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Abstract: In this paper, we define two different versions of intuitionistic fuzzy sylow p-subgroup of a finite group. Using these definitions the intuitionistic fuzzification of Sylow's Theorems are obtained. It is shown that all intuitionistic fuzzy sylow p-subgroups may not be conjugate of each other. We find the conditions when all intuitionistic fuzzy sylow p-subgroups will be conjugate of each other.

Keywords: Intuitionistic fuzzy set (IFS) , Intuitionistic fuzzy subgroup (IFSG), Flag ,Double pinned flags,Intuitionistic Fuzzy sylow p subgroup, (α, β) – cut, support of the intuitionistic fuzzy set.

Introduction

The converse of Lagrange's theorem is false: if G is a finite group and d|O(G), then there may not be a subgroup of G with order d. The simplest example of this is the group A_4 , of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [6] discovered that a converse result is true when d is a prime power: if p is a prime number and p^k |O(G) then G must contain a subgroup of order p^k . Sylow also discovered important relations among the subgroups with order the largest power of p dividing O(G), such as the fact that all subgroups of that order are conjugate to each other.
The notion of intuitionistic fuzzy subset was introduced by K.T. Atanassov [1] as a generalization of Zadeh's fuzzy sets [9]. In [2] R. Biswas introduce the concept of intuitionistic fuzzy subgroups of a group and studies some of its properties. After this many mathematician start intuitionistic fuzzification of others concept of group theory. In this paper, we put two distinct definitions of intuitionistic fuzzy sylow p- subgroup and study intuitionistic fuzzification of Sylow's Theorems.

Preliminaries

Definition 2.1[1]: Let X be a fixed non empty set. An intuitionistic fuzzy set (IFS) A of X is an object of the following form $A = \{< x, \mu_A(x), \nu_A(x) > : x \in X\}$ where $\mu_A: X \to [0,1]$ and $\nu_A: X \to [0,1]$ defined the degree of membership and degree of non membership of the element $x \in X$ respectively, for any $x \in X$,we have $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Proposition 2.2[1] Let $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ and $B = \{ < x , \mu_B(x) , \nu_B(x) > : x \in X\}$ be any two IFS's of X , then

(i) $A \subseteq B$ if and only if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ $\forall x \in X$

(ii) A = B if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$ $\forall x \in X$

(iii) $A \cap B = \{ < x, (\mu_A \cap \mu_B)(x) , (\nu_A \cap \nu_B)(x) > : x \in X\}$,
where $(\mu_A \cap \mu_B)(x) = \text{Min}\{ \mu_A(x) , \mu_B(x)\} = \mu_A(x) \wedge \mu_B(x)$
and $(\nu_A \cap \nu_B)(x) = \text{Max}\{ \nu_A(x) , \nu_B(x) \} = \nu_A(x) \vee \nu_B(x)$

(iv) $A \cup B = \{ < x, (\mu_A \cup \mu_B)(x) , (\nu_A \cup \nu_B)(x) > : x \in X\}$,
where $(\mu_A \cup \mu_B)(x) = \text{Max}\{ \mu_A(x) , \mu_B(x)\} = \mu_A(x) \vee \mu_B(x)$
and $(\nu_A \cup \nu_B)(x) = \text{Min}\{ \nu_A(x) , \nu_B(x) \} = \nu_A(x) \wedge \nu_B(x)$

Definition 2.3 [6]: Let A be an IFS of a universe set X . Then (α , β)-cut of A is a crisp $C_{\alpha , \beta}(A)$ of the IFS A is given by
$C_{\alpha , \beta}(A) = \{ x : x \in X$ such that $\mu_A(x) \ge \alpha , \nu_A(x) \le \beta \}$, where $\alpha , \beta \in [0,1]$ with $\alpha + \beta \le 1$.

Proposition (2.4) [6]: If A and B be two IFS's of a universe set X , then following holds

i. $C_{\alpha,\beta}(A) \subseteq C_{\delta,\theta}(A)$ if $\alpha \geq \delta$ and $\beta \leq \theta$
ii. $C_{1-\beta,\beta}(A) \subseteq C_{\alpha,\beta}(A) \subseteq C_{\alpha,1-\alpha}(A)$
iii. $A \subseteq B$ implies $C_{\alpha,\beta}(A) \subseteq C_{\alpha,\beta}(B)$
iv. $C_{\alpha,\beta}(A \cap B) = C_{\alpha,\beta}(A) \cap C_{\alpha,\beta}(B)$
v. $C_{\alpha,\beta}(A \cup B) \supseteq C_{\alpha,\beta}(A) \cup C_{\alpha,\beta}(B)$ equality hold if $\alpha + \beta = 1$
vi. $C_{\alpha,\beta}(\cap A_i) = \cap C_{\alpha,\beta}(A_i)$
vii. $C_{0,1}(A) = X$.

Definition 2.5: The support of the intuitionistic fuzzy set $A=\{< x, \mu_A(x), \nu_A(x) >: x \in X\}$ is the subset A of X defined by: $Supp_X(A) = \{ x \in A : \mu_A(x) > 0$ and $\nu_A(x) < 1\} = \cup \{C_{\alpha,\beta}(A) ; \alpha, \beta \in (0,1] , \alpha + \beta \leq 1\}$.

Definition 2.6[2]: An IFS $A = \{< x, \mu_A(x), \nu_A(x) >: x \in G\}$ of a group G is said to be intuitionistic fuzzy subgroup of G if

i. $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$
ii. $\mu_A(x^{-1}) = \mu_A(x)$
iii. $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$
iv. $\nu_A(x^{-1}) = \nu_A(x)$; for all $x , y \in G$.

Theorem (2.7)[6] : If A is IFS of a group G . Then A is IFSG of G if and only if $C_{\alpha,\beta}(A)$ is a subgroup of group G for all α , $\beta \in [0,1]$ with $\alpha + \beta \leq 1$, where $\mu_A(e) \geq \alpha$, $\nu_A(e) \leq \beta$ and e is the identity element of G.

Definition 2.7 [9] A flag is a maximal chain of subgroups of the form

$$G_0 \subset G_1 \subset G_2 \subset \ldots \subset G_n = G,$$

in which $G_0 = \langle e \rangle$ and all G_i's are called the components of the flag.

Definition 2.8[8] Consider a paired set of real numbers (α_i, β_i), such that $\alpha_i, \beta_i \in [0, 1]$ and $\alpha_i + \beta_i \leq 1$ for all $i = 0, 1, 2, ..., n$, then we call a chain $(\alpha_0, \beta_0) \geq (\alpha_1, \beta_1) \geq \ldots \geq (\alpha_n, \beta_n)$ a double keychain if and only if $1 = \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_n$ and $0 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_n$ and the pair (α_i, β_i) are called double pins.

Definition 2.9 [7] With the combination of flag and double keychain, we denote the chain $\langle e \rangle^{(1,0)} \subset G_1^{(\alpha 1,\beta 1)} \subset G_2^{(\alpha 2,\beta 2)} \subset \ldots \subset G_n^{(\alpha n,\beta n)}$ as double pinned flag.

The purpose of defining the double pinned flag is to define intuitionistic fuzzy sub-group in term of pinned flag. Corresponding to each pinned flag as defined above, we have an intuitionistic fuzzy subgroup $A = \{< x, \mu_A(x), \nu_A(x) >: x \in G\}$ in short $(\mu_A(x), \nu_A(x))$ as follows:

$$(\mu_A(x), \nu_A(x)) = \begin{cases} (1,0) & : \quad x \in \langle e \rangle \\ (\alpha_1, \beta_1) & : \quad x \in G_1 \setminus \langle e \rangle \\ (\alpha_2, \beta_2) & : \quad x \in G_2 \setminus G_1 \\ \ldots\ldots & \\ (\alpha_n, \beta_n) & : \quad x \in G_n \setminus G_{n-1} \end{cases}$$

The converse to the above result is also true

Intuitionistic Fuzzy Sylow p-subgroup

In this section, we introduce different definition of intuitionistic fuzzy sylow p-subgroup and obtained the intuitionistic fuzzification of the corresponding Sylow's Theorems.

Definition 3.1: An intuitionistic fuzzy subgroup A of G is called an intuitionistic fuzzy sylow p- subgroup if the support of A is sylow p- subgroup of G.

Definition 3.2: An IFSG A of finite group G is called an intuitionistic fuzzy sylow p subgroup of G if one of the $C_{\alpha,\beta}(A)$ is a Sylow p subgroup of G.
In definition (3,1), there may be two or more $C_{\alpha,\beta}(A)$ subgroups which are sylow p- subgroup.
But Support of A may not be sylow p-subgroup of G.
However definition (3.1) implies definition (3.2). Since in finite case support of IFSG A form one of the $C_{\alpha,\beta}(A)$ for some α, $\beta \in [0,1]$ such that $\alpha+\beta \le 1$.
Now, we give an example of intuitionistic fuzzy set A of a group G which is intuitionistic fuzzy sylow p-subgroup by definition (3.2) but not by definition (3.1).

Example 3.3: Let G = S_4, the symmetric group on four number 1, 2, 3 and 4. Define an IFS A of G by

$$\mu_A(x)=\begin{cases}1 & ; \quad if\ x\in\langle e\rangle\\ 0.5 & ; \quad if\ x\in\langle(123)\rangle\setminus\langle e\rangle\\ 0.3 & ; \quad if\ x\in A_4\setminus\langle(123)\rangle\\ 0 & ; \quad otherwise\end{cases} \quad \text{and} \quad \nu_A(x)=\begin{cases}0 & ; \quad if\ x\in\langle e\rangle\\ 0.3 & ; \quad if\ x\in\langle(123)\rangle\setminus\langle e\rangle\\ 0.5 & ; \quad if\ x\in A_4\setminus\langle(123)\rangle\\ 1 & ; \quad otherwise\end{cases}.$$

It can be easily verified that A is an IFSG of G. Note that $Supp_G(A) = A_4$.
But A_4 is not neither a sylow 2-subgroup of G nor a sylow 3-subgroup of G. Therefore, A is not an intuitionistic fuzzy sylow p-subgroup of G by definition (3.1).
But we notice that $C_{0.5,0.3}(A) = \langle (123) \rangle$ is a sylow 3-subgroup of G. Hence A is an intuitionistic fuzzy sylow 3-subgroup of G by definition (3.2).
We now illustrate some valid Sylow's theorem for the intuitionistic fuzzy set using definition (3.1).

Theorem 3.4 (Existence theorem): Let A be an IFSG of a finite group G of order $O(G) = p^k m$, where p is prime, k, m are positive integers with p, m are relatively prime . Assume that p divides the order of supp(A) = H. Then there exists an intuitionistic fuzzy sylow p- subgroup B of G such that $B \subseteq A$ in H.

Proof: If H (= supp(A)) is a Sylow p subgroup, then there is nothing to prove (as A itself be the sylow p subgroup of G such that $A \subseteq A$).
Let us suppose that H (= supp(A)) is not a Sylow p subgroup of G.
Let $\alpha = \min \{\mu_A(x): x \in G, \mu_A(x) > 0\}$ and $\beta = \max \{\nu_A(x): x \in G, \nu_A(x) < 1\}$
Clearly $\alpha \neq 0$ and $\beta \neq 1$.
Since G is finite , therefore, $C_{\alpha,\beta}(A) = supp_G(A) = H$.
Since p divides the order of H so by Sylow first theorem, there exists a sylow p subgroup of K of H. By our assumption the order of K must be p^t where $1 \le t \le k$.
Further K will be contained in Sylow p- subgroup K' of G.
We now define an intuitionistic fuzzy set B of G by

$$\mu_A(x)=\begin{cases}1 & ; \quad if\ x\in\langle e\rangle\\ \alpha & ; \quad if\ x\in K\setminus\langle e\rangle\\ \alpha-\varepsilon & ; \quad if\ x\in K'\setminus K\\ 0 & ; \quad if\ x\in G\setminus K'\end{cases} \quad \text{and} \quad \nu_A(x)=\begin{cases}1 & ; \quad if\ x\in\langle e\rangle\\ \beta & ; \quad if\ x\in K\setminus\langle e\rangle\\ \beta+\varepsilon' & ; \quad if\ x\in K'\setminus K\\ 1 & ; \quad if\ x\in G\setminus K'\end{cases}.$$

Where $0<\varepsilon<1$ and $\beta+\varepsilon' < 1$ for some $0 < \varepsilon' <1$.
It can be easily verified that B is an IFSG of G such that B ⊆A in H.
Also, Supp (B) = K' is a sylow p subgroup of G. Therefore, definition B is intuitionistic fuzzy sylow p- subgroup of G.

Definition 3.5: Two IFSs A and B of a group are said to be conjugate of each other if there exists some g ∈G such that B = $g^{-1}Ag$
i.e., $\mu_B(x) = \mu_A(g^{-1}xg)$ and $\nu_B(x) = \nu_A(g^{-1}xg)$; $\forall\ x\in G$

Proposition 3.6: Let A and B be two IFSGs of G having same double pinned flags. Then A and B are conjugate of each other if and only if $C_{\alpha,\beta}(B) = g^{-1} C_{\alpha,\beta}(A)\, g$ for some g ∈G, where
$\wedge(A) = \{ (\alpha_1,\beta_1), (\alpha_2,\beta_2)\ldots\ldots (\alpha_n,\beta_n)\}$ be the double pinned flags for IFSG A of G.

Proof: Firstly, let IFSGs A and B have same double pinned flags such that A and B are conjugate. Then by definition of conjugate intuitionistic fuzzy sets, there exists g ∈G such that

$$\mu_B(x)=\mu_A(g^{-1}xg) \quad and \quad \nu_B(x)=\nu_A(g^{-1}xg)\ ; \quad for\ x,g\in G.$$

$$\begin{aligned} Now, \quad g^{-1}C_{(\alpha_i,\beta_i)}(A)g &= g^{-1}\{x\in G:\ \mu_A(x)\ge\alpha_i \text{ and } \nu_A(x)\le\beta_i\}g \\ &=\{g^{-1}xg\in G:\ \mu_A(x)\ge\alpha_i \text{ and } \nu_A(x)\le\beta_i\} \\ &=\{t\in G:\ \mu_A(g^{-1}tg)\ge\alpha_i \text{ and } \nu_A(g^{-1}tg)\le\beta_i\} \\ &=\{t\in G:\ \mu_A(t)\ge\alpha_i \text{ and } \nu_A(t)\le\beta_i\} \\ &=C_{(\alpha_i,\beta_i)}(B). \end{aligned}$$

Conversely, let $g^{-1}C_{(\alpha_i,\beta_i)}(A)g = C_{(\alpha_i,\beta_i)}(B),$ for some g∈ G. To show that A and B are conjugate of each other. If possible let A and B are not conjugate of each other

$$\text{Let } \mu_B(x)>\mu_A(g^{-1}xg) \quad and \quad \nu_B(x)<\nu_A(g^{-1}xg); \quad for\ x,g\in G.$$

$$\text{Then,} \quad x\in C_{(\alpha,\beta)}(B) \quad and \quad g^{-1}xg\notin C_{(\alpha,\beta)}(A)$$

$$i.e., \quad x\in gC_{(\alpha,\beta)}(A)g^{-1} \quad and \quad g^{-1}xg\notin C_{(\alpha,\beta)}(A)$$

$$i.e., \quad g^{-1}xg\in C_{(\alpha,\beta)}(A) \quad and \quad g^{-1}xg\notin C_{(\alpha,\beta)}(A)$$

$$i.e., \quad (g^{-1}xg)^{-1}\in C_{(\alpha,\beta)}(A)\ and \quad g^{-1}xg\notin C_{(\alpha,\beta)}(A)$$

$i.e.,\ h^{-1}\in C_{(\alpha,\beta)}(A) \quad and \quad h\notin C_{(\alpha,\beta)}(A),$ which is not possible as $C_{(\alpha,\beta)}(A)$ is a subgroup of G.

Hence $\mu_B(x)=\mu_A(g^{-1}xg) \quad and \quad \nu_B(x)=\nu_A(g^{-1}xg); \quad for\ x,g\in G$
Hence , A and B are conjugate of each other.

Proposition 3.7: A conjugate of an intuitionistic fuzzy sylow p-subgroup of a group G is an intuitionistic fuzzy sylow p-subgroup of G.

Proof: Let B is a conjugate of an intuitionistic fuzzy sylow p- subgroup A of the group G
Then, by definition of conjugates there exists g ∈G such that $C_{\alpha,\beta}(B) = g^{-1} C_{\alpha,\beta}(A)\, g$; for all α, β
which implies that $Supp_G(B) = g^{-1}Supp_G(A)g$.
Since A is an intuitionistic fuzzy sylow p subgroup, therefore there is a sylow p- subgroup H of G contained in $Supp_G(A)$.
Now, $g^{-1}Hg$ being a conjugate of H is itself a sylow p subgroup of G (by Sylow second Theorem)
Further, $g^{-1}Hg$ is contained in $g\, Supp_G(A)\, g^{-1}$ (= $Supp_G(B)$) and so B is an intuitionistic fuzzy sylow p-subgroup of G.

The next example shows that converse is not true in general i.e., two distinct intuitionistic fuzzy sylow p subgroups need not to be conjugate to each other.

Example 3.8: Consider two Sylow 2- subgroups $H_1 = <(12),(1324)>$ $H_2 = <(13),(1234)>$ of S_4
We define two intuitionistic fuzzy subgroups A and B of S_4 by

$$\mu_A(x)=\begin{cases}1 & ;\ if\ x\in\langle e\rangle\\ 0.5 & ;\ if\ x\in\langle(12)(34)\rangle\setminus\langle e\rangle\\ 0.4 & ;\ if\ x\in\langle(1324)\rangle\setminus\langle(12)(34)\rangle\\ 0.3 & ;\ if\ x\in H_1\setminus\langle(1324)\rangle\\ 0 & ;\ otherwise\end{cases}\quad\text{and}\quad \nu_A(x)=\begin{cases}0 & ;\ if\ x\in\langle e\rangle\\ 0.3 & ;\ if\ x\in\langle(12)(34)\rangle\setminus\langle e\rangle\\ 0.4 & ;\ if\ x\in\langle(1324)\rangle\setminus\langle(12)(34)\rangle.\\ 0.5 & ;\ if\ x\in H_1\setminus\langle(1324)\rangle\\ 1 & ;\ otherwise\end{cases}$$

and

$$\mu_B(x)=\begin{cases}1 & ;\ if\ x\in\langle e\rangle\\ 0.5 & ;\ if\ x\in\langle(13)(24)\rangle\setminus\langle e\rangle\\ 0.4 & ;\ if\ x\in\langle(13),(24)\rangle\setminus\langle(13)(24)\rangle\\ 0.3 & ;\ if\ x\in H_1\setminus\langle(13),(24)\rangle\\ 0 & ;\ otherwise\end{cases}\quad\text{and}\quad \nu_B(x)=\begin{cases}0 & ;\ if\ x\in\langle e\rangle\\ 0.3 & ;\ if\ x\in\langle(13)(24)\rangle\setminus\langle e\rangle\\ 0.4 & ;\ if\ x\in\langle(13),(24)\rangle\setminus\langle(13)(24)\rangle.\\ 0.5 & ;\ if\ x\in H_1\setminus\langle(13),(24)\rangle\\ 1 & ;\ otherwise\end{cases}$$

Then it is easy to check that both A and B are intuitionistic fuzzy sylow 2- subgroups of G with
$Supp_G(A) = H_1$ and $Supp_G(B) = H_2$ respectively which are sylow 2- subgroups of G and so A and B are intuitionistic fuzzy sylow 2 subgroups of G.
But, A is not conjugate to B, for $C_{(0.4,0.5)}(A) = <(1324)>$ is cyclic, and $C_{(0.4,0.5)}(B) = <(13),(24)>$ is not cyclic.
However, we have following theorem on conjugacy whose proof is straight forward:

Theorem 3.9: If two intuitionistic fuzzy sylow p- subgroups A and B have the same double pinned flags and for each α, β ∈[0,1] , we have $C_{\alpha,\beta}(B) = g^{-1} C_{\alpha,\beta}(A)\, g$ for some fixed g ∈G, then they are conjugate of each other.

Conclusion

In this paper, we attempt to introduce the notion of intuitionistic fuzzy sylow p-subgroup of a finite group G and obtained some results which are different from the crisp sylow Theorems.

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